NORTHWESTERN UNIVERSITY

On Higher Genus Gromov-Witten Correspondences for Log Calabi-Yau Surfaces with Smooth Anticanonical Divisor

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

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EVANSTON, ILLINOIS

August 2025

Abstract

On Higher Genus Gromov-Witten Correspondences for Log Calabi-Yau Surfaces with Smooth Anticanonical Divisor

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Let X be a log Calabi-Yau surface with a smooth anti-canonical divisor E that is an elliptic curve. We prove an all genus correspondence and propose conjectures concerning the following enumerative theories associated to (X, E):

(1) the two-pointed log Gromov-Witten theory of (X, E) from the Gross-Siebert mirror symmetry program.

(2) additionally assuming X is toric, the open Gromov-Witten theory of special Lagrangians in the canonical bundle K_X, that is computed by the Topological Vertex.

(3) the closed Gromov-Witten theory of the projective compactification $\mathbb{P}(K_X \oplus \mathcal{O}_X)$.

When $X = \mathbb{P}^2$, we prove the conjectures in low degrees and all genus, and provide their computational validity in various cases.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor Eric Zaslow for his generosity and encouragement as a teacher. This thesis would not have been possible if it were not for our discussions, whereby through challenging my own assumptions, teaching me mathematics and physics, pointing out insightful computations to do, and introducing me to collaborators, he helped me to gradually find the way forward. Thank you for the time and patience you have committed as an advisor.

I thank Gus Schrader and Helge Ruddat for kindly agreeing to be on my thesis committee and reading this thesis.

I thank my collaborators Helge Ruddat and Tim Gräfnitz for sharing insightful ideas and discussing potential avenues in my research.

I thank Eric Zaslow, Helge Ruddat, Tim Gräfnitz, and Gus Schrader for reading drafts of my work.

I thank Emmy Murphy and Ezra Getzler for teaching me mathematics, having discussions, and being on my qualifying exam. I thank Aaron Naber for teaching me mathematics.

I thank Pierrick Bousseau and Hülya Arguz for their interest in my work, and inviting me to speak at the University of Georgia. I give special thanks to Pierrick for taking the time to meet and respond to my emails.

I thank Ursula Porod for guiding me as a teaching assistant.

I thank many people in graduate school for interesting discussions and good times, including but not limited to Colin Fan, Eivind Hjelle, Mingyuan Hu, Yuxuan Hu, Erik Hupp, Alex Karapetyan, Hy Lam, Wenyuan Li, Katy Loyd, Daniel Mallory, Anthony McCormick, Noah Riggenbach, Hanru Sun, Junxiao Wang, Ruoyu Wang, and Junho Won. I would like to thank the organizers of the Geometry/Physics seminar and the Mirror Symmetry and Mathematical Physics meetings.

I thank my Mom, Dad, Beth, Evan, and friends for their curiosity and support.

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CHAPTER 1

Introduction

1.1. Background and motivation

Gromov-Witten theory is motivated by some classical questions in enumerative geometry: what is the number N_d of degree d rational curves through $3d-1$ points in $\mathbb{CP}^2$? In the 1990s, Kontsevich and Manin found a recursive formula for all d [**KM94**], using Gromov-Witten theory,

$$N_d = \sum_{d_1+d_2=d, d_1,d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

Plugging in $N_1 = 1$, the first few numbers are $N_2 = 2, N_3 = 12, N_4 = 620, N_5 = 87304, N_6 = 26312976$... The first four numbers were known by the time of Zeuthen in the late 19th century, but computing N_5 and beyond is already quite difficult. Miraculously, Kontsevich's formula gives all of the numbers N_d. It is equivalent to the *WDVV equations* in *small quantum cohomology*.

Gromov-Witten (GW) invariants are rational numbers that are virtual counts of genus g curves in a target space X obtained by integrating incidence conditions on the moduli space of stable maps. There many techniques to compute them. When the target X carries a torus action $(\mathbb{C}^*)^n$, such as toric varieties, then *Atiyah-Bott localization* can be used [**K1**]. Another method known as *degeneration* deforms the target to another space X', with $X' \cong X$ in an appropriate sense, such that computing the Gromov-Witten invariants of X' is easier and equivalent to computing those of X [**Li**].

A jumpstart to the field came from studying the number of rational curves N_d in the quintic threefold $X = \{x^5+y^5+z^5+w^5+v^5 = 0\} \subseteq \mathbb{CP}^4$, which is an example of a *Calabi-Yau manifold*. The first few values of N_d are $N_1 = 2875, N_2 = 609250, N_3 = 317206375, N_4 = 242467530000$, and it is the content of the Clemens conjecture that $N_d < \infty$ for all d. Miraculously, an answer for all N_d came from topological string theory. Motivated by duality between Type IIA and Type IIB topological string theory, Candelas, de la Ossa, Greene and Parkes predicted the number of rational curves on the quintic threefold to all degrees d by computing period integrals on the *mirror*, which was the family of quintics $\check{X} = \{t(x^5+y^5+z^5+w^5+v^5)+xyzwv = 0 | t \in \mathbb{P}^1\} \subseteq \mathbb{C}P^4$ [**CdOGP**], [**GP**]. They were to able match known numbers. Their seminal work has since been proven to be correct by the work of Liu-Lian-Yau [**LLY**], Givental [**Giv**], and started the field of *mirror symmetry*.

When the target X is a point, Gromov-Witten theory reduces to computing intersection numbers on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$, which was formulated in Witten's Conjecture [**Wit**] and proven by Kontsevich [**K2**]. These invariants are governed by the KdV hierarchy. Two Fields Medals were won by Kontsevich and Mirzirkhani each for their work on computing intersection numbers of $\overline{\mathcal{M}}_{g,n}$, [**K2**], [**Mir**].

The Gromov-Witten theory of a curve such as $\mathbb{P}^1$ has been computed by Okounkov and Pandharipande using degeneration and Hurwitz theory, and they show that the 2-Toda hierarchy governs the Gromov-Witten theory of $\mathbb{P}^1$ [**OP**]. The Gromov-Witten theory of an elliptic curve was computed by Okounkov and Pandharipande, and shown to exhibit quasimodular properties [**OP**].

For K3 surfaces X, its genus 0 Gromov-Witten theory was first prediced by Yau and Zaslow, who used string duality between Type II string theory on K3 and heterotic string on T^4 [**YZ**]. The genus 0 Gromov-Witten invariants of $K3$ are expressed by the weight 12 modular discriminant. A proof of the Yau-Zaslow conjecture was given in symplectic geometry by [**BL**]. In differential geometry, Taubes showed a correspondence between Seiberg-Witten invariants of 4-manifolds and certain Gromov-Witten invariants [**Tau**].

For Calabi-Yau 3-folds X, Maulik, Nekrasov, Okounkov, and Pandharipande conjectured that Gromov-Witten invariants are equivalent to Donaldson-Thomas invariants obtained from the moduli space of ideal sheaves associated to curves [**MNOP**]. Their conjecture was proven when X is toric [**MOOP**]. Techniques such as mirror symmetry and Eynard-Orantin recursion [**FLZ**], [**FRZZ**] or the Topological Vertex and large N duality [**AKMV**] can be used to compute Gromov-Witten invariants of toric Calabi Yau 3-folds.

Gromov-Witten invariants are called *closed* when the curves one is counting are closed. When they have boundary, they are called *open* Gromov-Witten invariants. Much of the work in open Gromov-Witten theory has been done in symplectic geometry from studying J-holomorphic curves from Riemann surfaces with boundary. *Local* Gromov-Witten invariants concern situations when the target X may be embedded into an ambient space Y, and the Gromov-Witten invariants of Y may be computed from that of X. One may also study curves with prescribed tangencies to a divisor D in the target X. Such invariants are called *relative* Gromov-Witten invariants. When the divisor has normal crossings singularity, *logarithmic* Gromov-Witten invariants are used; log geometry tells us that a normal crossings divisor is log smooth. It is of interest to establish correspondences between the different kinds of Gromov-Witten invariants, and sometimes these correspondences have their origin in dualities from physics. Gromov-Witten invariants are conjectured to satisfy beautiful summation formulas in terms of integer *Gopakumar-Vafa or BPS invariants.*

1.2. Enumerative invariants from log Calabi-Yau surfaces

We now turn to the setting of the thesis. Let (X,E) be a log Calabi-Yau surface, i.e. X is a smooth projective surface, with a anticanonical divisor $E = E_1 + \ldots + E_l \in |-K_X|$ that is possibly nodal. The pair (X,E) carries a rich enumerative theory, and from it one can associate many invariants. Assuming $l \geq 2$, the work of [**BBvG**] establishes the equivalence of open and closed Gromov-Witten theories, log Gromov-Witten theory of (X,E), and quiver Donaldson-Thomas theory associated from (X,E). Log Calabi-Yau surfaces are of interest from the viewpoint of mirror symmetry as T-duality, where special Lagrangian fibrations are constructed in the complement $X \setminus E$ [**Aur**]

Now, suppose that E is a smooth anticanonical divisor, and we will interchangeably refer to X as a Fano surface. By the adjunction formula, E is an elliptic curve. Let K_X the canonical bundle of X, and let $Z := \mathbb{P}(K_X \oplus \mathcal{O}_X)$ be its projective compactification. Let $\pi : \hat{X} \to X$ be the blow up of X at a point. Let $R_{g,2}(X(\log E))$ be the genus g, two-pointed, logarithmic Gromov-Witten invariant of X with λ_g-insertions with two prescribed tangencies to E, one fixed and the other varying. Let $O_g(K_X)$ be the genus g, winding 1 open Gromov-Witten invariant of K_X, and $N_{g,1}(Z)$ be a genus g, closed Gromov-Witten invariant of Z passing through one point.

When $g = 0$, we have the following triangle of equalities (up to multiplication by rational constants),

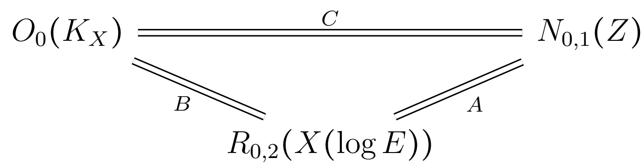


Figure 1.1. A triangle of genus 0 equalities (up to rational constants) between open, log, and closed invariants associated from (X, E).

The equality A was recently established by [**Wan**] with a degeneration argument. The equality B is an *open-log* correspondence established in [**GRZ**]. The techniques used there were not limited to *tropical curve counting*, and *scattering diagrams* from the *Gross-Siebert mirror symmetry program*. The equality C is an *open-closed* correspondence that was established by [**Cha**] assuming X is toric.

1.3. Main results

The subject of this thesis is to extend the triangle of results in Figure 1.1 to higher genus. We show that the relationship is not as simple in $g > 0$, and the equalities become modified by the Gromov-Witten theory of E and two-pointed logarithmic invariants of (X, E).

1.3.1. Higher genus local Gromov-Witten invariants from projective bundles - Extending A

To extend equality A to higher genus, we apply the degeneration formula [**KLR**] to the degeneration that was considered in [**Wan**]. Then, we use the higher genus log-local principle of [**BFGW**] to establish an all genus correspondence between invariants of Z and local Gromov-Witten invariants of $\hat{X}$,

Theorem 1. *There exists constants* $c(g,\beta) \in \mathbb{Q}$ *such that,*

$$\sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} N_{g,1}(Z, \beta + h)\hbar^{2g} Q^\beta = \sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} \left[c(g,\beta) n_g \left(K_{\hat{X}}, \pi^*\beta - C\right) \left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}}} \right)^{2g-2} Q^\beta \right] - \Delta^{pl}$$

where $\mathbf{q} = e^{i\hbar}$*, and* $n_g(K_{\hat{X}}, \pi^*\beta - C)$ *is the genus* g*, Gopakumar-Vafa invariant of* $K_{\hat{X}}$ *in curve class* $\pi^*\beta - C$*. The discrepancy term* Δ^{pl} *is a function of the stationary Gromov-Witten theory of* E *and two-pointed log invariants of* $X(\log E)$*.*

Using the invariance of Gromov-Witten invariants under simple flops of 3-folds [**LR**], we prove an all genus blow up formula for the invariants $N_{g,1}(Z)$,

Theorem 2. *Let* $c(g,\beta) \in \mathbb{Q}$ *and* Δ^{pl} *be as in Theorem 1, and let* $W = Bl_p Z$ *be the blow up of* Z *at a point* p *on the infinity section of* Z*. Then, we have that,*

$$\sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} N_{g,1}(Z, \beta + h)\hbar^{2g} Q^\beta = \sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} \left[c(g,\beta) N_{g,0}(W, \beta + \tilde{L})\hbar^{2g} Q^\beta \right] - \Delta^{pl}$$

We provide explicit formulas of Theorems 1, 2 in genus 1.

1.3.2. Higher genus open-log conjecture for smooth divisor with results for $\mathbb{P}^2$ - Extending B

To extend equality B to higher genus, we follow the roadmap of [**GRZ**] and use the scattering diagrams of Gross-Siebert to first prove an all genus correspondence between $\mathbf{q}$-refined tropical curves and local invariants of $\hat{X}$,

Theorem 3. *Let* (X, E) *be a log Calabi-Yau surface* X *with smooth anticanonical divisor* E*. Then, we have,*

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} R^{trop}_{g,(e-1,1)}(X,\beta)\hbar^{2g}Q^{\beta} = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\left[(-1)^{\beta\cdot E+g-1}n_g(K_{\hat{X}},\pi^*\beta - C)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}}}\right)^{2g-2}Q^{\beta}\right] - \Delta^{ol}$$

where $R^{trop}_{g,(e-1,1)}(X,\beta)$ *is the genus* g*, two-legged,* $\mathbf{q}$*-refined tropical curve count in the scattering diagram associated to* (X,E) *(see Chapter 3 for definition of these invariants), and* $n_g(K_{\hat{X}},\pi^*\beta - C)$ *is the genus* g*, Gopakumar-Vafa invariant of* $K_{\hat{X}}$ *in class* $\pi^*\beta - C$*,* Δ^{ol} *is a discrepancy term defined in Equation 6.10 that is a function of the stationary Gromov-Witten theory of* E *and two pointed log invariants of* $X(\log E)$*, and* $\mathbf{q} = e^{i\hbar}$*.*

From Theorem 3, we conjecture an all genus correspondence between two-pointed log invariants of (X, E) and open invariants of K_X,

Conjecture 1 (Open-log conjecture for smooth divisor)**.** *Let* (X, E) *and* Δ^{ol} *be as in Theorem 3. Furthermore, assume that* X *is toric, and* $\pi : \hat{X} \to X$ *is a toric blow up. Then, we conjecture the following correspondence,*

$$\sum_{\substack{\beta\in H_2^+(X,\mathbb{Z}),\\ g\geq 0}} (e-1)R_{g,(e-1,1)}(X(\log E),\beta)\hbar^{2g}Q^{\beta} = \sum_{\substack{\beta\in H_2^+(X,\mathbb{Z}),\\ g\geq 0}}\left[\frac{(-1)^{g+1}}{(e-1)}n_g^{open}(K_X,\beta+\beta_0,1)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}}}\right)^{2g-2}Q^{\beta}\right] - \Delta^{ol}$$

where $R_{g,(e-1,1)}(X(\log E),\beta)$ *are two-pointed log invariants of* $X(\log E)$ *with* λ_g*-insertion in class* β*, and* $n_g(K_X,\beta+\beta_0,1)$ *is the genus* g*, winding 1, framing 0, open BPS invariant of a single outer AV-brane* L *in* K_X*, and* $\mathbf{q} = e^{i\hbar}$*.*

We provide explicit formulas of Conjecture 1 in genus 1 and 2, and provide computational validity.

When $X = \mathbb{P}^2$, we use the topological vertex (Theorem 6) to prove Conjecture 1 in low degrees and all genus,

Theorem 4. *Let* $X = \mathbb{P}^2$ *and* $H \in H_2(\mathbb{P}^2,\mathbb{Z})$ *the hyperplane class. Then Conjecture 1 holds in curve classes* $\beta = dH$ *for* $d = 1,2,3,4$ *and all genus.*

We discuss applications of Conjecture 1 and Theorem 4 to quantum theta functions and open mirror symmetry in Section 6.6, as part of upcoming work [**GRZZ**].

1.3.3. Higher genus open-closed conjecture with results for $\mathbb{P}^2$ - Extending C

Using Theorem 1, we conjecture that,

Conjecture 2 (Open-closed conjecture for projective bundles)**.** *Let* $c(g,\beta) \in \mathbb{Q}$ *and* Δ^{pl} *be as in Theorem 1. Furthermore, assume that* X *is toric, and* $\pi : \hat{X} \to X$ *is a toric blow up. Define* $d(g,\beta) := (-1)^{g+1}c(g,\beta)$*. We conjecture the following equality,*

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g}Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\left[d(g,\beta)n_g^{open}(K_X,\beta+\beta_0,1)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}}}\right)^{2g-2}Q^\beta\right]-\Delta^{pl}$$

where $n_g^{open}(K_X, \beta + \beta_0, 1)$ *is the genus g, 1-holed, winding 1, open BPS invariant in curve class* $\beta + \beta_0$ *of an outer Aganagic-Vafa brane L in framing 0 (see Chapter 4 for its definition).*

We provide explicit formulas of Conjecture 2 in genus 1 and 2.

When $X = \mathbb{P}^2$, we use the topological vertex (Theorem 24) to prove Conjecture 1 in low degrees and all genus,

Theorem 5. *Let* $X = \mathbb{P}^2$ *and* $H \in H_2(\mathbb{P}^2, \mathbb{Z})$ *the hyperplane class. Then Conjecture 2 holds in curve classes* $\beta = dH$ *for* $d = 1, 2, 3, 4$ *and all genus.*

1.3.4. Higher genus open-closed BPS conjecture for toric Calabi-Yau threefolds with results for local $\mathbb{P}^2$

By comparing known open and closed Gopakumar-Vafa invariants [**GZ**], [**MV**], [**HKR**], [**KZ**], we make the following conjecture relating open and closed Gopakumar-Vafa invariants of the toric Calabi-Yau threefolds that are K_X and $K_{\hat{X}}$,

Conjecture 3 (Open-closed BPS conjecture for toric Calabi-Yau threefolds)**.** *Let X be a toric del Pezzo surface, and* $\pi : \hat{X} \to X$ *a toric blow up with exceptional curve C. Then we have the following equality,*

$$n_g(K_{\hat{X}}, \pi^*\beta - C) = (-1)^{g+1} n_g^{open}(K_X, \beta + \beta_0, 1)$$

where $n_g(K_{\hat{X}}, \pi^*\beta - C)$ *is the genus g, closed Gopakumar-Vafa invariant of the canonical bundle* $K_{\hat{X}}$ *in curve class* $\pi^*\beta - C$*, and* $n_g^{open}(K_X, \beta + \beta_0, 1)$ *be the genus g, 1-holed, winding*

1, open BPS invariant of K_X with boundary on a single, outer Aganagic-Vafa brane in framing 0 in disc class $\beta + \beta_0 \in H_2(K_X, L)$.

We use the topological vertex [**AKMV**] and its refined version [**IKV**] to show that,

Theorem 6. *Conjecture 3 is true for $X = \mathbb{P}^2$ in curve classes $\beta = dH$ for $d = 1, 2, 3, 4$ and in all genus.*

1.4. Roadmap of the thesis

In Chapter 2, we present preliminaries. In Chapter 3, we describe scattering and its quantized version, define **q**-refined tropical curve invariants, and describe Gross-Siebert mirror symmetry applied to $\mathbb{P}^2$. In Chapter 4, we define the Gromov-Witten invariants needed for the main results of the thesis. In Chapter 5, we prove Theorems 1 and 15, state Conjecture 2, and prove Theorem 5. In Chapter 6, we prove Theorem 3, state Conjecture 1, prove Theorem 4, and discuss applications to [**GRZZ**]. In Chapter 9, we state Conjecture 3 and prove Theorem 6. In Appendix A, we summarize the $g > 0$ log-local principle of [**BFGW**], derive its form in genus 1 and 2, specialize it to $\mathbb{P}^2$, and also describe an alternative way to compute an invariant from Chapter 5.

CHAPTER 2

Preliminaries

2.1. Gromov-Witten Theory

2.1.1. The moduli space of stable maps

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of nodal, genus g curves with n distinct marked points. The moduli space can be defined with geometric invariant theory, and we refer to [**HM**] for details. When $g > 1$, $\overline{\mathcal{M}}_{g,n}$ is a non-singular Deligne-Mumford stack of dimension $3g - 3 + n$. Examples include $\overline{\mathcal{M}}_{0,n} = (\mathbb{P}^1)^{n-3}$, and $\overline{\mathcal{M}}_{1,1}$ is parametrized by the j-line.

Let X be a smooth projective variety.

Definition 1. *A stable map to X is the following data:*

(1) $(C, p_1, \ldots, p_n, f)$ is an at worst nodal curve C of arithmetic genus g with n distinct smooth points $p_1, \ldots, p_n$ of C and a morphism $f : C \to X$ such that $f_*[C] = \beta \in H_2(X, \mathbb{Z})$. We say β is the curve class of f.

(2) The map f is stable or has finite automorphism group, where two stable maps $f : (C, p_1, \ldots, p_n) \to X$ and $f' : (C', p_1', \ldots, p_n') \to X$ are isomorphic if there exists an isomorphism $\varphi : (C, p_1, \ldots, p_n) \to (C', p_1', \ldots, p_n')$ such that $\varphi(p_i) = p_i'$ and $f' \circ \varphi = f$. Equivalently if f is constant on a component of C, then if the genus of the component is 0, it is required to have at least 3 points that are either marked points or nodes. If the genus of the component is 1, it is required to have at least 1 point that is either a marked point or node.

Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the moduli space of genus g, n-marked stable maps to X in the curve class β. Elements are isomorphism classes $[f : C \to X]$, which we abbreviate as $[f]$.

Example 1. $\overline{\mathcal{M}}_{g,n}(X,0) \cong \overline{\mathcal{M}}_{g,n} \times X$. *When the curve class is 0, then stable maps are just constant maps.*

Example 2. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^n,1)$ *is the Grassmannian of lines* $Gr(1,n)$.

Example 3. *The moduli space* $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2,2)$ *is the space of complete conics. There are four components of the moduli space, containing maps of the following kind,*

(1) Dense open set corresponding to parametrizations of embedded, nonsingular, irreducible conics.

(2) A stable map mapping from two $\mathbb{P}^1$*'s joined together by a node to a reducible conic consisting of two lines meeting transversely.*

(3) A stable map of the same domain in 2) mapping to a single line together with a specified point that is the image of the node.

(4) Double covers of $\mathbb{P}^1$ *by* $\mathbb{P}^1$.

The moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is compact. There are evaluation maps $ev = ev_1 \times \ldots \times ev_n : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X^n$ defined by $[f] \mapsto (f(p_1),\ldots,f(p_n))$. For each marked point p_i, there is a forgetful map $ft_i : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ that forgets the i-th marked point and stabilizes, as long as the domain and codomain moduli spaces exist. The universal curve $\mathcal{U}$ of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ can be identified with $\overline{\mathcal{M}}_{g,n+1}(X,\beta)$, and we have the diagram,

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & & \end{array}$$

where the universal map f can be identified with the evaluation map at the $(n+1)$-marked point, and π can be identified with ft_{n+1}.

2.1.2. Virtual fundamental class

We would like to study enumerative geometry on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ by pulling back cohomology classes in $H^*(X)$ and integrating. To do so, we must construct a well-defined *virtual fundamental class* in $A_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$.

When $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is smooth and compact, then the fundamental class is the virtual fundamental class, and we may define Gromov-Witten invariants by integrating cohomology classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$. However, in general, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ can be quite ill-behaved: it can be reducible, non-reduced, or of impure dimension.

We want to calculate the expected or virtual dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ by analyzing its deformation/obstruction theory. Let $[f : C \to X]$ be a stable map. Its deformations consist of deformations of the map or of the domain curve C. Deformations of the map are given by the group $H^0(C, f^*TX)$, and obstructions are given by $H^1(C, f^*TX)$. Deformations of domain curve is of dimension $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$. Therefore, the (complex) virtual dimension is given by,

$$\text{vdim } \overline{\mathcal{M}}_{g,n}(X,\beta) = h^0(C, f^*TX) - h^1(C, f^*TX) + 3g - 3 + n$$

By the Riemann-Roch formula, the above is,

$$\begin{aligned} \text{vdim } \overline{\mathcal{M}}_{g,n}(X,\beta) &= \deg(f^*TX) + \text{rk}(f^*TX)(1-g) + 3g - 3 + n \\ &= (\dim X - 3)(1-g) + \int_\beta c_1(TX) + n \end{aligned} \tag{2.1}$$

Notice the formula simplifies considerably when X is a Calabi-Yau 3-fold, i.e. when $c_1(TX) = 0$.

Example 4. *Here is an example of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ having impure dimension. Let $H \in H_2(\mathbb{P}^2,\mathbb{Z})$ be the hyperplane class of $\mathbb{P}^2$, and consider $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3H)$. It has 3 disjoint components:*

(1) Maps from nonsingular genus 1 curves to cubics

(2) Maps from an elliptic curve with one rational tail that contract the elliptic curve and is a degree 3 map from $\mathbb{P}^1$

(3) Maps from an elliptic curve with two rational tails that contracts the elliptic curve and maps to a line and a conic.

The dimension of 1) is the virtual dimension, which is 9. For 2), consider the following diagram,

$$\begin{array}{ccc} \overline{\mathcal{M}}_{1,1}(\mathbb{P}^2,0) \times_{\mathbb{P}^2} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2,3H) & \longrightarrow & \overline{\mathcal{M}}_{1,1}(\mathbb{P}^2,0) \\ \downarrow & & \downarrow{\scriptstyle ev} \\ \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2,3H) & \xrightarrow{ev} & \mathbb{P}^2 \end{array}$$

The dimension of 2) is the dimension of the fibre product, which is 10. For 3), the maps from the rational tails are in the fiber product $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^2,2) \times_{\mathbb{P}^2} \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2,1)$, which has dimension 7. After adding the dimension of $\overline{\mathcal{M}}_{1,2}$ to account for the contracted elliptic curve, the dimension is 9. We see that $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2,3H)$ is reducible and impure.

There is a case when $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is well-behaved,

Definition 2. *We say X is convex if $H^1(C, f^*TX) = 0$ for all $f \in \overline{\mathcal{M}}_{g,n}(X,\beta)$.*

When X is convex, $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a smooth Deligne-Mumford stack. Examples of convex varieties include homogeneous spaces such as projective space, flag varieties, and Grassmannians. When the moduli space is unobstructed, the virtual fundamental class is equal to the fundamental class. In general, when the moduli space is smooth but not of the expected dimension, the obstruction vector bundle Ob over $\overline{\mathcal{M}}_{g,n}(X,\beta)$ (with fiber given by $H^1(C, f^*TX)$) has constant rank, and the virtual fundamental class is given by,

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = e(Ob) \cap [\overline{\mathcal{M}}_{g,n}(X,\beta)]$$

Let $\mathfrak{M}_{g,n}$ be the Artin stack of prestable curves and $p : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$ the morphism that forgets the map, only remembers the domain curve and does not stabilize. Consider the complex,

$$E^{\bullet}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}} = (R^{\bullet}\pi^* f^*TX)^{\vee}$$

There is a morphism $E^{\bullet}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}} \to \mathbb{L}^{\bullet}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}}$, which is a perfect relative obstruction theory for the map p. The moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a proper Deligne-Mumford stack for all g and n. By the work of [**BF**], there is a virtual fundamental class,

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in A_{\text{vdim}}(\overline{\mathcal{M}}_{g,n}(X,\beta), \mathbb{Q})$$

We refer to Section 2.2.2 for more details about virtual fundamental classes and obstruction theories.

2.1.3. Gromov-Witten invariants

Let $\gamma_i \in H^*(X, \mathbb{Q})$. The genus g, n-pointed, Gromov-Witten invariant of class β corresponding to the γ_i is defined to be

$$N_{g,n}(X, \beta; \gamma_1 \otimes \ldots \otimes \gamma_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1^* \gamma_1 \wedge \ldots ev_n^* \gamma_n$$

We will also write $N_{g,n}(X, \beta; \gamma_1 \otimes \ldots \otimes \gamma_n)$ as $\langle \gamma_1, \ldots, \gamma_n \rangle^X_{g,n,\beta}$.

Notice that $N_{g,n}(X, \beta; \gamma_1 \otimes \ldots \otimes \gamma_n) = 0$ if $\sum_i 2 \deg_{\mathbb{C}} \gamma_i \neq \text{virdim } \overline{\mathcal{M}}_{g,n}(X, \beta)$.

We mention here some important properties of Gromov-Witten invariants. See [**KM94**], [**Lee**] for more details.

(1) (Divisor axiom): Suppose $\gamma_n \in H^2(X, \mathbb{Q})$, then,

$$N_{g,n}(X, \beta; \gamma_1 \otimes \ldots \otimes \gamma_n) = (\int_\beta \gamma_n) N_{g,n-1}(X, \beta; \gamma_1 \otimes \ldots \otimes \gamma_{n-1})$$

(2) (Fundamental axiom): Let $1 \in H^0(X)$ be the identity in cohomology. Then $N_{g,n}(X, \beta; \gamma_1 \otimes \ldots \otimes 1) = 0$. Equivalently, this conditions can be formulated on virtual classes as for $1 \leq i \leq n+1$,

$$[\overline{\mathcal{M}}_{g,n+1}(X, \beta)]^{vir} = ft_i^* [\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$$

(3) (Point mapping axiom): For $g = 0, \beta = 0$, we have $N_{g,n}(X, \beta; \gamma_1 \otimes \ldots \otimes \gamma_n) = \int \gamma_1 \cup \gamma_2 \cup \gamma_3$ when $n = 3$, and 0 otherwise.

2.1.3.1. Tautological Classes. There are natural characteristic classes one has on $\overline{\mathcal{M}}_{g,n}$. Let $\mathbb{E}$ be the *Hodge bundle* which is a vector bundle over $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose fiber above $[f : C \to X]$ is given by $H^0(C, K_C)$, or the vector space of holomorphic differentials on C. Since $\mathbb{E}$ is bundle of rank g, we define the λ_i-classes as,

$$\lambda_i := c_i(\mathbb{E}), \quad 1 \le i \le g \tag{2.2}$$

There are no holomorphic differentials for a genus 0 curve, and we define $\lambda_0 := 1$. Mumford's relation states that $c(\mathbb{E})c(\mathbb{E}^\vee) = 1$. In particular, $\lambda_g^2 = 0$ for all $g > 0$.

Let $\mathbb{L}_i$ be a line bundle over $\overline{\mathcal{M}}_{g,n}(X, \beta)$ whose fiber above $[f : (C, p_1, \ldots, p_n) \to X]$ is the cotangent line $T^*_{p_i}C$. The ψ-classes are defined as,

$$\psi_i := c_1(\mathbb{L}_i), \quad 1 \le i \le n$$

2.1.3.2. Descendant invariants. Gromov-Witten invariants with ψ- or λ-classes present, i.e.

$$\int_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \prod_{i=1}^{n} ev_i^*(\gamma_i) \cup \psi_i^{a_i} \cup \prod_{j=1}^{g} \lambda_j^{b_j}$$

are called *Hodge integrals*. If the $b_j = 0$ for all j, then they are called *gravitational descendants* or *descendant invariants*. They are called *stationary* if the $\gamma_i \in H^*(X, \mathbb{Q})$ are point classes. Classically, when $X = pt$, then we have Witten's conjecture which states that,

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{a_1} \dots \psi_n^{a_n} = \binom{n-3}{a_1, \dots, a_n}$$

Descendant invariants satisfy the following equations, s

(1) (String equation)

$$\langle \psi_1^{a_1}\gamma_1, \dots, \psi_n^{a_n}\gamma_n, 1\rangle_{g,n+1,\beta}^X = \sum_{i=1}^n \langle \psi_1^{a_1}\gamma_1, \dots, \psi_i^{a_i-1}\gamma_{i-1}, \psi_i^{a_i-1}\gamma_i, \psi_i^{a_i+1}\gamma_{i+1} \dots, \psi_n^{a_n}\gamma_n\rangle_{g,n,\beta}^X$$

(2) (Dilaton equation)

$$\langle \psi_1^{a_1}\gamma_1, \dots, \psi_n^{a_n}\gamma_n, \psi_{n+1}1\rangle_{g,n+1,\beta}^X = (2g-2+n)\langle \psi_1^{a_1}\gamma_1, \dots, \psi_n^{a_n}\gamma_n\rangle_{g,n,\beta}^X$$

2.2. Intersection theory

We provide some notions of intersection theory that we will use. The main reference is [**Ful**].

Let X be an algebraic variety. A *cycle* is a finite, formal sum of irreducible subvarieties of X with integer coefficients. A cycle is called *k-dimensional* if it consists of k-dimensional subvarieties. Define $Z_k(X)$ be the group of k-dimensional cycles.

Suppose that V is a subvariety of $X \times \mathbb{P}^1$. The subgroup $Rat(X)$ of *rationally equivalent classes* is generated by cycles of the form $[V_0] - [V_\infty]$, where V_i denotes the subvariety of V restricted to the fiber above $t \in \mathbb{P}^1$. The *Chow group* $A_*(X) := Z(X)/Rat(X)$ is the group of rational equivalence classes of cycles in X. The Chow group is graded by the dimension of subvarieties, so we may write,

$$A_*(X) = \oplus_k A_k(X)$$

We write $\alpha \sim \beta$ to mean two cycles α and β are rationally equivalent, and $[\alpha]$ to be the rational equivalence class of α.

The *Chow ring* $A^*(X) = \oplus_k A^k(X)$ is defined by $A^k(X)$, or the group of codimension k cycles of X. The multiplication is given by an intersection product that is defined as,

$$A^i(X) \otimes A^j(X) \to A^{i+j}(X)$$

$$[\alpha] \cdot [\beta] \mapsto [\alpha \cap \beta]$$

The multiplication is well-defined due to the fact that if the intersection is not proper, there exists a rationally equivalent cycle $\alpha' \sim \alpha$ such that $\alpha' \cdot \beta$ is proper.

Suppose that $X \subset Y$ is a closed subscheme with defining ideal I. The *normal cone* $C_X Y$ of X in Y is defined as,

$$C_X Y := \operatorname{Spec}\left(\sum_{n=0}^{\infty} I^n / I^{n+1}\right)$$

The normal sheaf of X in Y is defined as,

$$N_{X/Y} := \operatorname{Spec}(Sym^{\bullet}(I/I^2))$$

When X is regularly embedded in Y, then $C_X Y$ is a vector bundle. If there exist only linear relations among elements in I, then the normal cone is the normal sheaf. Recall that I/I^2 is the conormal sheaf of X in Y.

We define a proper pushforward of cycles. Suppose that $f : X \to Y$ is a proper morphism. For any subvariety $V \subset X$, the image $W = f(V)$ is a subvariety of Y. Then, the *proper pushforward* is defined as $f_*[V] = deg(V/W)[W]$ where $deg(V/W) = [R(V) : R(W)]$ if $\dim W = \dim V$ or else 0.

2.2.1. Gysin homomorphisms

The main reference if [**Ful**], Chapter 2. Let $D \subset X$ be an effective Cartier divisor, and $\alpha \in Z_k X$ a k-cycle. Let $j : \alpha \hookrightarrow X$ be the inclusion. We may define the intersection product $D \cdot \alpha$ as the Weil divisor class of $[j^* D]$, which has support in $A_{k-1}(|D| \cap \alpha)$. We have *Gysin pullback* maps $i^* : Z_k X \to A_{k-1} D$ defined by $i^*(\alpha) = D \cdot \alpha$.

Suppose we have a regular embedding $i : X \hookrightarrow Y$ of codimension d, a morphism $f : Y' \to Y$, and the following fibre square,

$$\begin{array}{ccc} X' & \xrightarrow{h} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

The normal cone $C_{X'/Y'}$ is a closed subcone of $g^* N_X Y$. We define *refined Gysin homomorphisms*,

$$i^! : Z_k Y' \to A_{k-d} X'$$

$$\sum_i n_i [V_i] \mapsto \sum_i n_i (X \cdot V_i)$$

They are given explicitly by the following composition of maps,

$$i^! : Z_k(Y') \xrightarrow{\sigma} Z_k(C_{X'/Y'}) \hookrightarrow A_k(g^* N_X Y) \xrightarrow{s^*} A_{k-d}(X')$$

where the *specialization homomorphisms* $\sigma : Z_k Y \to Z_k(C_{X/Y})$ are defined by $\sigma[V] \mapsto [C_{V\cap X} V]$, the middle map is induced by inclusion, and s^* is flat pullback by the 0-section X' in $g^* N_X Y$. On cycles, the map $i^!$ will send $[V] \in Z_k(Y')$ to $[C_{V\cap X'} V] \in A_k(g^* N_X Y)$, then intersect with the 0-section X' in $g^* N_X Y$. When $Y' = Y$ and $f = Id_Y$, the homomorphisms are simply called the *Gysin homomorphisms* $i^* : A_k(Y) \to A_{k-d}(X)$. We remark that if $V \subset W$, then $i^!([V]) \subset h^{-1}(W)$, by Cartesian-ness of the diagram.

We have the following useful theorems related to pullbacks,

Theorem 7. *(Theorem 3.3 of* [**Ful**]*) Let $\pi : E \to X$ be a rank r vector bundle, then the flat pullback $\pi^* : A_{k-r}(X) \to A_k(E)$ is an isomorphism for all k.*

Theorem 8 (Projection formula)**.** *Let $f : X \to Y$ be a proper morphism. Then,*

$$f_*(f^*\alpha \cap \beta) = \alpha \cap f_*\beta$$

We also have the following excision sequence of Chow groups,

Lemma 1. *Let Y be a closed subscheme of a scheme X, and let $U = X \smallsetminus Y$. Let $i : Y \to X$, $j : U \to X$ be the inclusions. Then the sequence*

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j_*} A_k U \to 0$$

is exact for all k.

Proof. See Proposition 1.8 of [**Ful**]. □

We present the excess intersection formula. Suppose that we have the fibre diagram,

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

where i and i' are regular embeddings of codimension d and d' with normal bundles N and N', respectively. Let $E := g^*N/N'$ be the quotient bundle of rank $d - d'$ on X'.

Theorem 9. *For any $\alpha \in A_k(Y'')$, we have*

$$i^!\alpha = c_{d-d'}(q^*E) \cap i'^!\alpha$$

in $A_{k-d}(X'')$.

As a corollary, suppose we have fibre diagram,

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

with i' an isomorphism, then we have

$$i^!\alpha = c_d(g^*N) \cap \alpha$$

Specializing to $X' = Y' = X$, we have the classical self-intersection formula,

$$i^*i_*\alpha = c_d(N) \cap \alpha$$

for all $\alpha \in A_*(X)$.

2.2.2. Virtual Fundamental Classes

2.2.2.1. Motivation. Virtual fundamental classes allow one to do intersection theory on moduli spaces that are not smooth or of pure dimension, and to account for anomalies such as non-transverse intersections or non-transverse tangency conditions. If one considers the classical enumerative question of finding the number of plane conics tangent to five lines, a naive answer of 32 can be obtained initially by counting solutions to linear equations. However one must account for degenerate contributions from the non-transverse locus of double lines in order to get the correct answer 1.

The correct number can be computed with *Segre classes*. One finds the correct number of conics by computing a certain class related to the Segre class of the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$, which is the locus of double lines. Virtual fundamental classes in some sense generalize Segre classes used in the classical story to more general settings.

2.2.2.2. Obstruction Theories. We first define obstruction theories, in order to give a definition of virtual fundamental classes.

Let $f : X \to Y$ be a morphism of DM-type between algebraic stacks. Let $\mathbb{L}^\bullet_{X/Y}$ be the relative cotangent complex. If f is in fact a regular embedding of schemes, then $\mathbb{L}^\bullet_{X/Y}$ is given by $[I/I^2 \to 0]$, where I is the defining ideal of $X \subset Y$. If $Y = pt$, then $L^\bullet_{X/pt} = \Omega^1_X$, or the sheaf of Kähler differentials.

Definition 3. *Let $f : X \to Y$ be a morphism of DM-type between algebraic stacks. A relative perfect obstruction theory $E^\bullet_{X/Y}$ on X is a two-term complex of coherent sheaves $E^\bullet = [E^{-1} \to E^0] \in D^bCoh(X)$ in perfect amplitude $[-1,0]$, together with a map $E^\bullet_{X/Y} \to \mathbb{L}^\bullet_{X/Y}$, that is an isomorphism on h^0 and a surjection on h^{-1}.*

Example 5. *Recall the moduli space of stable maps* $\overline{\mathcal{M}}_{g,n}(X,\beta)$*. Let* $\pi : \mathcal{U} \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ *be the universal curve and* $f : \mathcal{U} \to X$ *be the universal map. Let* $\rho : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$ *be the morphism to the Artin stack of prestable curves that forgets the map. Define* $E^{\bullet}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}} := (R^{\bullet}\pi_* f^* TX)^{\vee}$*. It is a relative perfect obstruction theory for* ρ*.*

We say that the obstruction theory is *trivial* if $E^{\bullet}_{X/Y} \cong L^{\bullet}_{X/Y}$. Suppose that we have morphisms $X \to Y \to Z$ with $Y \to Z$ étale, then there is an isomorphism of perfect relative obstruction theories $E_{X/Y} \cong E_{X/Z}$.

2.2.2.3. Virtual Pullbacks. Virtual pullbacks were introduced in [**Man08**] and apply in more general settings than flat or Gysin pullbacks.

The virtual pullback is defined with respect to an obstruction theory $E^{\bullet}_{X/Y}$. Suppose that we have an embedding $X \hookrightarrow Y$, and a vector bundle $E^{\bullet}_{X/Y}$ of rank r with an embedding of cones over X, $C_{X/Y} \hookrightarrow E^{\bullet}_{X/Y}$. We have the following composition of maps,

$$A_*(Y) \xrightarrow{\sigma} A_*(C_{X/Y}) \xrightarrow{i_*} A_k(E^{\bullet}_{X/Y}) \xrightarrow{0^!_{E^{\bullet}_{X/Y}}} A_{*-r}(X)$$

where σ is the specialization homomorphism, and $0^!_{E^{\bullet}_{X/Y}}$ is the inverse of flat pullback. The virtual pullback $f^!_{E^{\bullet}_{X/Y}}$ is the composition of the above maps. Note the similarity of the definition of virtual pullbacks with that of refined Gysin homomorphisms. Indeed, the latter are a special version of the former: when $f : X \to Y$ is a smooth morphism of schemes, the normal cone $C_{X/Y}$ is a smooth vector bundle, and the virtual pullback $f^!_{C_{X/Y}} : A_*(Y) \to A_*(X)$ agrees with flat pullback of cycles. Like Gysin pullbacks, virtual pullbacks also commute with proper pushforward.

We are now ready to define virtual fundamental classes, as the intersection of the 0-section of an obstruction bundle $E^\bullet_{X/Y}$ with the normal cone $C_{X/Y}$. Suppose that Y is pure dimensional, so that the fundamental class $[Y]$ is connected. We define the virtual fundamental class of X with respect to the obstruction theory $E^\bullet_{X/Y}$ to be the class,

$$[X]^{vir}_{E^\bullet_{X/Y}} := f^!_{E^\bullet_{X/Y}}[Y] = 0^!_E[C_{X/Y}]$$

where $0^!_E$ denotes the inverse of the flat pullback $\pi : E_{X/Y} \to X$.

Example 6. *Let $X \hookrightarrow Y$ be an embedding, with Y pure dimensional. Then, the obstruction bundle $E_{X/Y}$ is a vector bundle with an embedding of cones $C_{X/Y} \hookrightarrow E_{X/Y}$. When $X \hookrightarrow Y$ is regular, then $C_{X/Y}$ is the normal vector bundle. There is an exact sequence of vector bundles,*

$$0 \to N_{X/Y} \to E_{X/Y} \to E \to 0$$

where the vector bundle E is defined by the sequence and called the excess bundle. The virtual class is given by,

$$[X]^{vir}_{E_{X/Y}} = c_{top}(E) \cap [X]$$

Virtual fundamental classes are compatible under base change: suppose that we have the fiber diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

where f is an embedding of codimension d. The refined Gysin map $f^! : A_*(Y') \to A_{*-d}(X')$. It is shown in [**LT**], Proposition 3.9 that

$$[X']^{vir} = f^![Y']^{vir}$$

2.2.2.4. Intrinsic normal cone. We make a remark about the construction of the intrinsic normal cone in [**BF**]. Suppose that we have local charts U_i on X which embed into ambient spaces W_i. We have normal cones C_{U_i/W_i} for each i. There exists an action of $TW_i|U_i$ that preserves C_{U_i/W_i}, and therefore we have a stack

$$\mathfrak{C}_X|U_i \cong C_{U_i/W_i}/TW_i|_{U_i}$$

The stack $\mathfrak{C}_X$ is called the intrinsic normal cone. It is called intrinsic because its definition does not actually depend on the embeddings into W_i. A perfect obstruction theory $E_X^\bullet$ is equivalent to the data of local charts $U_i \subset X$ and embeddings $U_i \subseteq W_i$ with W_i smooth, with obstruction bundles E_{U_i/W_i} such that the tangent-obstruction complexes $[TW_i|_{U_i} \to E_{U_i/W_i}]$ glue to form $E_X^\bullet$. It gives rise to a global stack $\mathfrak{E}_X := h^1/h^0(E_X^\bullet)$ that is locally defined as $h^1/h^0(E_X^\bullet)|_{U_i} \cong E_{U_i/W_i}/TW_i|_{U_i}$. The rank of $\mathfrak{E}_X$ is $rk(h^1(E_X^\bullet)) - rk(h^0(E_X^\bullet))$. The condition for perfectness guarentees that $\mathfrak{E}_X$ contains the intrinsic normal cone $\mathfrak{C}_X$. We refer to [**BF**], Section 2 for more details.

Remark 1. *We shall use the notation $[X, E^\bullet]$ from* [**BF**] *to mean the virtual class on X given by the perfect obstruction theory $E^\bullet$.*

2.3. Toric Geometry

Toric varieties provide many useful examples of log schemes and log Calabi-Yau varieties. In this section, we review some basic notions of toric geometry from [**Ful**].

Let $M \cong \mathbb{Z}^n$ be a lattice, and $N = Hom_{\mathbb{Z}}(M, \mathbb{Z})$ the dual lattice. Denote $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Suppose that $\sigma \subset M_{\mathbb{R}}$ is a strictly convex, rational polyhedral cone, i.e. a cone such that $\sigma \cap -\sigma = 0$. Define the dual cone $\check{\sigma}$ to be the set

$$\check{\sigma} = \{n \in N_{\mathbb{R}} | \langle n, m \rangle \geq 0, \forall m \in \sigma\}$$

Definition 4. *The affine toric variety X_σ associated to the cone $\sigma \subset N_{\mathbb{R}}$ is the set,*

$$X_\sigma := \operatorname{Spec} k[\check{\sigma} \cap N]$$

Definition 5. *A fan Σ is a collection of strongly convex, rational cones in $N_{\mathbb{R}}$ such that 1) each face of a cone in Σ is a cone in Σ and 2) the intersection of two cones in Σ is a face in each.*

A toric variety X_Σ is formed from a fan Σ, by appropriately gluing along faces, i.e. if $\tau = \sigma_1 \cap \sigma_2$ is the intersection of two cones σ_1 and σ_2, then X_{σ_1} and X_{σ_2} are glued along the common open subset X_τ. In particular, every fan contains the cone $\{0\}$, which gives the torus $T \cong (\mathbb{C}^*)^n$. The one dimensional cones $\Sigma^{[1]}$ in the fan Σ correspond to codimension 1, T-invariant subvarieties of X.

We call the union of the divisors corresponding to the 1-dimensional cone the *toric boundary*, and denote it by ∂X_Σ, i.e.

$$\partial X_\Sigma = \cup_{\rho\in\Sigma[1]} D_\rho$$

It is the complement of the big torus orbit. The divisor $\sum_{\rho\in\Sigma[1]} D_\rho$ is anti-canonical (can show it is rationally equivalent to the divisor associated to the holomorphic volume form $d(\log x_1)\wedge\ldots\wedge d(\log x_n)$. Hence, a toric variety with its toric boundary is naturally a log scheme and a log Calabi-Yau variety.

Example 7. *Consider the fan Σ in $\mathbb{R}^2$ whose one dimensional cones are given by $\mathbb{R}_{\geq 0}(1,0), \mathbb{R}_{\geq 0}(0,1)$, and $\mathbb{R}_{\geq 0}(-1,-1)$. By analyzing the gluing of affine toric charts, one see that this describes the fan of the projective plane $\mathbb{P}^2$. Adding the ray $\mathbb{R}_{\geq 0}(0,-1)$ or the sum of the two rays $\mathbb{R}_{\geq 0}(-1,-1)+\mathbb{R}_{\geq 0}(1,0)$, corresponds to blowing up a toric fixed point of $\mathbb{P}^2$. The resulting fan Σ' is that of the first Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1})$.*

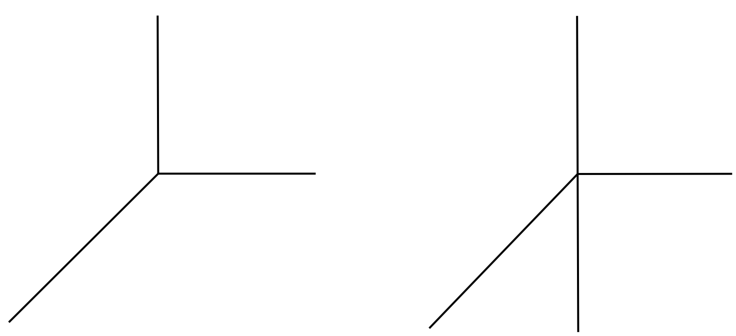


Figure 2.1. The fan of $\mathbb{P}^2$ on the left and fan of $\mathbb{F}_1$ on the right.

There is a useful characterization of smoothness of toric varieties. A toric varietiy X_Σ is smooth if and only if the intersection of each cone $\sigma\in\Sigma$ with $N\cong\mathbb{Z}^n$ gives a $\mathbb{Z}$-basis. A toric variety X_Σ is compact if and only if $|\Sigma| = \mathbb{R}^n$.

A *refinement* Σ' of a fan Σ is a fan whose cones are contained in cones of Σ, and the supports are equal $|\Sigma'| = |\Sigma|$. A refinement Σ' gives a proper birational map of toric varieties $X_{\Sigma'} \to X_\Sigma$, and we have the following theorem,

Theorem 10 ([**Ful**]). *There exists a refinement Σ' of any fan Σ such that $X_{\Sigma'} \to X_\Sigma$ is a resolution of singularities.*

In symplectic geometry, a manifold M is called toric if $Diff(M)$ carries a Hamiltonian action of the algebraic torus $(\mathbb{C}^*)^n$. Such an action endows M with a moment map $\mu : M \to \mathfrak{g} \cong \mathbb{R}^n$. By Delzant's theorem, the image of μ is a convex polytope whose vertices are images of fixed points of the torus action. The dimensions of the polytope are determined by a polarization or choice of symplectic form on M.

2.4. Tropical Geometry

Tropical geometry was introduced as a more combinatorial, piecewise linear approach to algebraic geometry. One works over the tropical semiring. Many classical theorems in algebraic geometry such as Bezout's theorem for intersection numbers have tropical versions.

Let Γ be a connected graph. Let $\Gamma^{[0]}$ be the set of 0-dimensional vertices of h, $\Gamma^{[1]}$ be the set of bounded edges, and $\Gamma^{[1]}_\infty$ be the set of noncompact, unbounded edges of h. Let $w : \Gamma^{[1]} \to \mathbb{N}$ be a function that assigns a non-negative integer weight to each edge $E \in \Gamma^{[1]}$. We will sometimes write the weight of an edge E as w_E.

Definition 6. *A parametrized tropical curve is a map $h : \Gamma \to \mathbb{R}^2$ such that,* *(1) If $E \in \Gamma^{[1]}$ and $w(E) = 0$, then $h|_E$ is constant. Otherwise, $h|_E$ is a proper embedding of E into a line of rational slope in $M_{\mathbb{R}}$.*

(2) Let $V \in \Gamma^{[0]}$, and let $E_1, \ldots, E_n$ be the edges adjacent to V. Let $m_i \in M$ be the primitive tangent vector $h(E_i)$ pointing away from $h(V)$. Then, we have,

$$\sum_{i=1}^{n} w(E_i) m_i = 0$$

We say that a tropical curve is in the toric variety X_Σ if its unbounded, noncompact edges are translates of the one dimensional cones in $\Sigma^{[1]}$.

Note that the loops of tropical curves must be contracted, else the map $h : \Gamma \to \mathbb{R}^2$ would not be an embedding. The weights of non-compact edges are intersection numbers of the curve with the toric divisors of X_Σ.

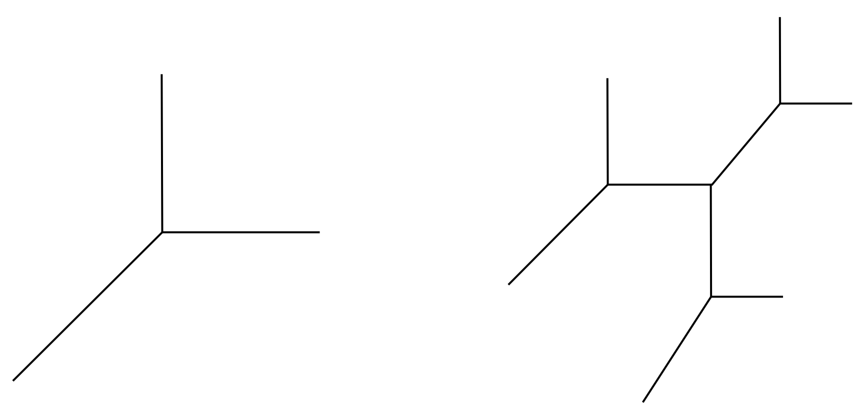


Figure 2.2. A degree 1 tropical curve (left) and degree 2 tropical curve (right) in $\mathbb{P}^2$.

Definition 7. *Let T_Σ be the free abelian group generated by the one cones $\Sigma^{[1]}$ of the fan. For $\rho \in \Sigma^{[1]}$, denote by $t_\rho \in T_\Sigma$ the corresponding generator.*

If h is a tropical curve in X_Σ, the degree of h is $\Delta(h)$ defined by,

$$\Delta(h) = \sum_{\rho \in \Sigma^{[1]}} d_\rho t_\rho$$

where d_ρ *is the number of edges* $E \in \Gamma_\infty^{[1]}$ *with* $h(E)$ *a translate of* ρ. *Denote* $|\Delta(h)| = \sum_{\rho \in \Sigma^{[1]}} d_\rho$.

Definition 8. *A tropical disc* $h : \Gamma \to \mathbb{R}^2$ *is a tropical curve with a choice of univalent vertex* V_∞*, adjacent to a unique compact edge* E_∞*. In addition,* h *satisfies the balancing condition for all vertices* $V \neq V_\infty$*. The edge* E_∞ *need not be parallel to any translate of* $\rho \in \Sigma^{[1]}$.

Definition 9. *The Maslov index* $MI(h)$ *of a tropical disc* $h : \Gamma \to \mathbb{R}^2$ *is,*

$$MI(h) := 2|\Delta(h)|$$

where $|\Delta(h)|$ *is computed without counting the edge* E_∞.

Definition 10. *Given a tropical curve* $h : \Gamma \to \mathbb{R}^2$*, and a trivalent vertex* $V \in \Gamma^0$*, let* m_i *and* m_j *be distinct vectors that are any two of the three primitive outgoing vectors at* $h(V)$*. Define*

$$m_V := \det(m_i | m_j)$$

By the balancing condition at V*, the number* m_V *is well-defined, and is called the multiplicity of the vertex* V*. The multiplicity of an edge* E *of weight* w_E *is defined as,*

$$m_E := \frac{(-1)^{w_E+1}}{w_E^2}$$

The multiplicity of h is defined as,

$$m_h = \frac{1}{|Aut(h)|} \prod_{V \in \Gamma^{[0]}} m_V \prod_{E \in \Gamma^{[1]}} m_E$$

Block and Göttsche defined a refinement of the classical multiplicity of tropical curves [**BG**]. Let $\mathbf{q} = e^{i\hbar}$ be a formal variable for "quantization". Their definition has subsequently been used to relate $\mathbf{q}$-refined tropical curve counts to higher genus logarithmic Gromov-Witten invariants.

Definition 11 ($\mathbf{q}$-multiplicity of tropical curves)**.** *Given a tropical curve* $h : \Gamma \to \mathbb{R}^2$*, and a trivalent vertex* $V \in \Gamma^0$*, define the* $\mathbf{q}$*-multiplicity of* V *to be the expression,*

$$m_V(\mathbf{q}) = \frac{\mathbf{q}^{m_V/2} - \mathbf{q}^{-m_V/2}}{\mathbf{q}^{1/2} - \mathbf{q}^{-1/2}}$$

where m_V *is the classical multiplicity of* V *in Definition 10. The* $\mathbf{q}$*-multiplicity of an edge* E *of weight* w_E *is defined as,*

$$m_E(\mathbf{q}) := \frac{(-1)^{w_E+1}}{w_E} \frac{\mathbf{q}^{1/2} - \mathbf{q}^{-1/2}}{\mathbf{q}^{w_E/2} - \mathbf{q}^{-w_E/2}}$$

The $\mathbf{q}$*-multiplicity of h is defined as,*

$$m_h(\mathbf{q}) = \frac{1}{|Aut(h)|} \prod_{V \in \Gamma^{[0]}} m_V(\mathbf{q}) \prod_{E \in \Gamma^{[1]}} m_E(\mathbf{q})$$

Remark 2. *The weight of edges is sometimes ignored when defining the* $\mathbf{q}$*-multiplicity of tropical curves, as is the case in* [**Bou2**], [**BG**].

Remark 3. *The expression for the multiplicity of tropical curves is motivated by the proof of Mikhalkin's genus 0 correspondence between counts of complex algebraic curves in toric surfaces and counts of tropical curves with multiplicity in* $\mathbb{R}^2$*. For a tropical curve* h*, the argument in* [**Mik**] *associates* m_h *number of log curves to it. Nishinou-Siebert generalized* [**Mik**] *to toric varieties of arbitrary dimensions* [**NS**]*. Bousseau genearlized* [**Mik**] *to higher genus* [**Bou2**]

2.5. Log Geometry

Let X be a scheme. Logarithmic geometry was first introduced by Illusie-Fontaine, and Kato, to handle open and possibly singular varieties. Log Calabi-Yau spaces (X, D) are naturally amenable to log geometry as $X \setminus D$ is Calabi-Yau. Log geometry was also introduced into Gromov-Witten theory by Gross and Siebert, in order to develop a relative theory for when the divisor is singular. Endowing source curves and target spaces with log structures allows the possibility of negative tangency orders. Log geometry is especially useful when one has a degeneration to a normal crossings variety. Such a degeneration is log smooth despite the existence of the central singular fiber. In addition, nodal curves are log smooth. We review some of the basic notions from log geometry that we will use.

Definition 12. *A pre-log structure is a sheaf of monoids* $\mathcal{M}_X$ *on* X *with a morphism of monoids* $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$*, where the monoid structure on* $\mathcal{O}_X$ *is given by multiplication of functions. A log structure is a pre-log structure satisfying* $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$*.*

Example 8. *When* $\mathcal{M}_X = \mathcal{O}_X^*$ *and* α_X *is the inclusion, we have the trivial log structure.*

Example 9. *Let* $i : D \hookrightarrow X$ *be a possibly singular divisor. Then the divisorial log structure* $\mathcal{M}_{(X,D)}$ *is defined as*

$$\mathcal{M}_{(X,D)} \coloneqq i_* \mathcal{O}^*_{X \smallsetminus D} \cap \mathcal{O}_X$$

or the regular functions on X *that are invertible away from* D*. On open sets away from* D*, the sheaf* $\mathcal{M}_{(X,D)}$ *is isomorphic to* $\mathcal{O}^*_X$*.*

Example 10. *Let* Q *be a monoid, and consider* $(\operatorname{Spec} k, Q \oplus k^*)$*, where* k *is a field. Let* $\alpha_X(x,q) : \operatorname{Spec} k \oplus Q \to k^*$ *be given by*

$$\alpha_X(x,q) = \begin{cases} x & q = 0 \\ 0 & q \neq 0 \end{cases}$$

This defines a log structure, and $(\operatorname{Spec} k, Q \oplus k^*)$ *is called the standard log point. We will write the standard log point with monoid* Q *as* pt_Q*.*

Definition 13. *A log scheme is a scheme* X *with a log structure* $\mathcal{M}_X$*.*

Notation 1. *We will often write the log scheme* X *with divisorial log structure* D *as* $X(\log D)$.

The *characteristic or ghost sheaf* $\overline{\mathcal{M}}_X$ is defined as $\overline{\mathcal{M}}_X \coloneqq \mathcal{M}_X / \mathcal{O}^*_X$. A log structure $\mathcal{M}_X$ is called *saturated* if each of its stalks $\mathcal{M}_{X,x}$ is saturated as a monoid, i.e. if it is integral and whenever $p \in \mathcal{M}^{gp}_{X,x}$ with $mp \in \mathcal{M}_{X,x}$, then $p \in \mathcal{M}_{X,x}$. The log structure is *fine*, if locally on the log scheme, it is isomorphic to the pullback of the divisorial log structure of a (generalized) toric variety $\operatorname{Spec}\mathbb{Z}[P]$. We say that a log scheme is fs if it is fine and saturated.

Definition 14. *A morphism of logarithmic schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a morphism of a schemes, together with a morphism of sheaves of monoids $f^\#$ such that the following diagram is commutative,*

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{f^\#} & B \\ \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle \alpha_X} \\ \mathcal{O}_Y & \xrightarrow{f^*} & \mathcal{O}_X \end{array}$$

Here f^* is pullback of functions.

The notion of log smoothness is defined similarly as with schemes, with the infinitesimal lifting criterion.

Definition 15. *Let $f : X \to Y$ be a morphism of fine log schemes. It is called log smooth (respectively étale) if the underlying morphism of schemes is locally of finite presentation and for any commutative diagram,*

$$\begin{array}{ccc} B_0 & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle f} \\ B_1 & \xrightarrow{\psi} & Y \end{array}$$

étale locally on B_1, there exists a (respectively there exists a unique) morphism $g : B_1 \to X$ such that $\varphi = g \circ j$ and $\psi = f \circ g$. The map j is a strict closed immersion, and its closed subscheme is defined by an ideal J with $J^2 = 0$.

A morphism of log schemes $f : X \to Y$ is called *strict* if $f^{-1}\overline{\mathcal{M}}_Y \cong \overline{\mathcal{M}}_X$.

Definition 16. *The logarithmic tangent bundle* TX^{log} *is defined as the sheaf of derivations on* X *that also preserve the ideal sheaf of* D*. We sometimes write* TX^{log} *as* $TX(-\log D)$*.*

Example 11. *When* X *is a toric variety, we have* $TX^{log} \cong \mathcal{O}_X^{\dim X}$*.*

Suppose that X is a log scheme with normal crossings divisor D. Then D is locally defined by the equation $x_1 \dots x_r = 0$, and the logarithmic tangent bundle is locally generated by the derivations $x_1\frac{\partial}{\partial x_1}, \dots, x_r\frac{\partial}{\partial x_r}, x_{r+1}, \dots, x_n$.

The log tangent bundle fits into the exact sequence,

$$0 \to TX(-\log D) \to TX \to ND \to 0$$

We see that on D, the log tangent bundle agrees with the tangent bundle of D. Away from D, it is isomorphic to the tangent bundle of X.

The sheaf of logarithmic 1-forms $\Omega^{1,log}$ is defined as the subsheaf of $j_*\Omega^1_{X\setminus D}$ (with $j : X \setminus D \hookrightarrow X$) that is locally generated by $\frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r}, x_{r+1}, \dots, x_n$. We have the relation,

$$TX^{log} = Hom_{\mathcal{O}_X}(\Omega_X^{1,log}, \mathcal{O}_X)$$

2.5.1. Log curves

Definition 17. *A logarithmic or log curve is a logarithmically smooth and flat morphism of fs log schemes* $\pi : X \to S$ *such that all geometric fibers are reduced and connected schemes of pure dimension 1 that satisfies the following condition: if* $U \subset C$ *is the non-singular locus of* π *then there exist sections* $x_1, \dots, x_n$ *of* π *such that,*

$$\overline{\mathcal{M}}_C|_U \cong \pi^*\overline{\mathcal{M}}_S \oplus \bigoplus_{i=1}^{n}(x_i)_*\mathbb{N}$$

Kato provided a classification of log smooth curves. Suppose that $f : C \to X$ is log smooth morphism of fine saturated log schemes, with $X = \operatorname{Spec} A$ where $(A, \mathfrak{m})$ is a complete local ring. Let $0 \in X$ be the closed point, and let Q be the monoid $\overline{\mathcal{M}}_{X,0}$. There is a chart $\sigma : Q \to A$ defining the log structure on X. Let C_0 be the fibre of f over $0 \in W$. Then, etale locally at $x \in C_0$, the log scheme C is isomorphic to one of the following three cases:

(1) $S = \operatorname{Spec} A[u]$, where the log structure is induced by the chart,

$$Q \to \mathcal{O}_S, \quad q \mapsto f^*\sigma(q)$$

(2) $S = \operatorname{Spec} A[u,v]/(uv - t)$ for some $t \in \mathfrak{m}$, where the log structure is induced by the chart,

$$\mathbb{N}^2 \oplus_{\mathbb{N}} Q \to \mathcal{O}_S, \quad ((a,b),q) \mapsto u^a v^b f^*\sigma(q)$$

The fibred sum is defined by the diagonal map $\mathbb{N} \to \mathbb{N}^2$ and $\mathbb{N} \to Q$ is a homomorphism determined by f given by $1 \mapsto \alpha \in Q$ with $\sigma(\alpha) = t$.

(3) $S = \operatorname{Spec} A[u]$ with the log structure induced by the chart,

$$\mathbb{N} \oplus Q \to \mathcal{O}_S, \quad (a,q) \mapsto u^a f^*\sigma(q)$$

In case (1), the log structure on C is just smooth pullback of the log structure on X. In case (2), the curve C is nodal and (a,b) encodes the vanishing orders at the node. In case (3), the $\mathbb{N}$-summand comes from the stalk of a standard log point, which is thought

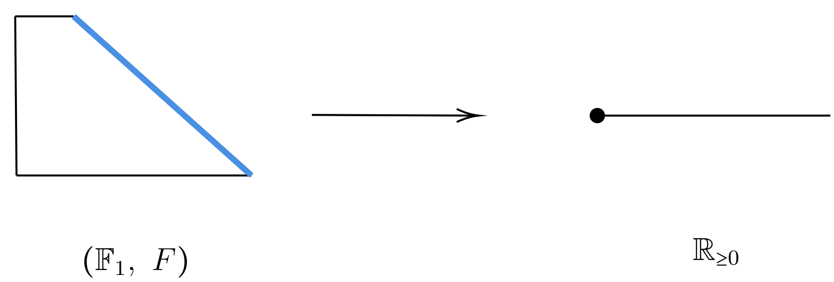


Figure 2.3. The log scheme $\mathbb{F}_1$ with divisorial log structure given by a fiber F (left) and its tropicalization $\mathbb{R}_{\geq 0}$ (right).

of as a marked point on C or a section $pt_{\mathbb{N}} \to C$. The log structure on C is the sum of the pullback log structure from X and the divisorial log structure given by the marked point.

2.5.2. Tropicalization

Tropicalization is a procedure that constructs a cone complex from a log scheme.

Definition 18. *Given a log scheme X, its tropicalization is defined as,*

$$Trop(X) := \amalg_{x \in X} (Hom(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}))/\sim$$

where the equivalence relation is generated by dualizing generization maps $\overline{\mathcal{M}}_{X,x} \to \overline{\mathcal{M}}_{X,x'}$ when x is a specialization of x'.

Example 12. *Let X be a toric variety with toric boundary ∂X as divisorial log structure. Its tropicalization is isomorphic to the fan Σ of X as generalized cone complexes.*

Example 13. *The tropicalization of the standard log point $(\operatorname{Spec} k, Q \oplus k^*)$ is given by the cone $\mathbb{R}_{\geq 0}$.*

Example 14. *Let* $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$ *be the first Hirzebruch surface, and* F *be a* $\mathbb{P}^1$*-fiber of* $\mathbb{F}_1$ *when it is considered as a* $\mathbb{P}^1$*-bundle. Let* $(\mathbb{F}_1, F)$ *be the log scheme* $\mathbb{F}_1$ *with* F *as divisorial log structure. Then, the tropicalization of* $(\mathbb{F}_1, F)$ *is given by a single cone* $\mathbb{R}_{\geq 0}$*. See Figure 2.3.*

We note that the tropicalization does not come with an embedding to $N_{\mathbb{R}}$.

Tropicalization is a covariant functor from the category of log schemes to the category of cone complexes. Suppose that we have a stable log map,

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow \pi & & \downarrow \\ pt_{\overline{\mathcal{M}}} & \longrightarrow & S \end{array}$$

where pt_Q is the standard log point associated to a constant monoid sheaf Q. Tropicalization turns the stable log map into a diagram of cone complexes,

$$\begin{array}{ccc} \Sigma(C) & \xrightarrow{\Sigma(f)} & \Sigma(X) \\ \downarrow \Sigma(\pi) & & \downarrow \\ \mathbb{R}_{\geq 0} & \longrightarrow & \Sigma(S) \end{array}$$

When X is a toric variety, $\Sigma(\pi)$ is a family of parametrized tropical curves mapping to the fan Σ of X. The fiber over the origin $0 \in \mathbb{R}_{\geq 0}$ is obtained by contracting the dual graph of C to a graph with a unique vertex. The domain graphs of the tropical curves are the dual intersection graphs of the stable log map.

2.6. Logarithmic Gromov-Witten Theory

Logarithmic Gromov-Witten invariants were introduced in relative Gromov-Witten theory to deal with possibly normal crossings divisors. They play a key role in Gross-Siebert mirror symmetry for log Calabi-Yau surfaces.

2.6.1. Stable log maps

Definition 19. *A stable log map determines a commutative diagram in the category of log schemes,*

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ W & \longrightarrow & S \end{array}$$

where π is a log smooth, proper integral curve. For each geometric point $w \to W$, the restriction of f onto the fiber C_w is an ordinary stable map.

Definition 20. *A class β of stable log maps to X is the following,*

(1) The data $\overline{\beta}$ of an underlying ordinary stable map, i.e. the genus g of C, the number n of marked points, and data A boudning the degree as described in [**BF**], *pg.12.*

(2) Strict closed embeddings $Z_1, \ldots, Z_n \subset X$, together with sections $s_i \in \Gamma(Z_i, (\overline{\mathcal{M}}^{gp}_{Z_i})^*)$.

We have a natural morphism of log structures $f^\flat : f^*\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$.

Definition 21. *A stable log map $(C/W, x_1, \ldots, x_n, f)$ is of class β if the underlying ordinary stable map is of type (g, n, A) and if for any i, we have $im(f \circ x_i) \subset Z_i$, and for any geometric point $w \to W$, the map,*

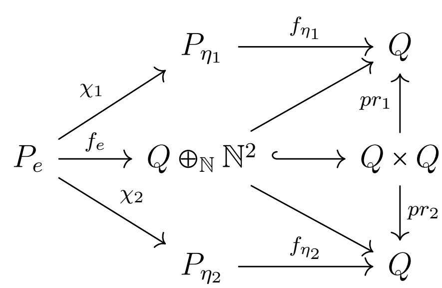
$$u_w : \overline{\mathcal{M}}_{Z_i, f(x_i(w))} = (f^*\overline{\mathcal{M}}_X)_{x_i(w)} \xrightarrow{f^\flat} \overline{\mathcal{M}}_{C, x_i(w)} = \overline{\mathcal{M}}_{W,w} \oplus \mathbb{N} \xrightarrow{pr_2} \mathbb{N}$$

One should think of the map u_w as encoding the contact order of the marked point x_i.

2.6.2. Moduli space of stable log maps

The moduli stack of stable log maps $\overline{\mathcal{M}}(X)$ is perhaps too large if arbitrary log structures on the base W are allowed. Gross and Siebert consider a specific open substack $\mathcal{M}(X) \subset \overline{\mathcal{M}}(X)$ of *basic* stable log maps. Basicness is a condition that guarantees properties of $\mathcal{M}(X)$ such as quasi-compactness, algebraicity, finite type, and stable reduction.

Consider a stable log map $C \to X$ over $W = pt$. Let $Q := \overline{\mathcal{M}}_W$ and $P_x := \overline{\mathcal{M}}_{X,f(x)}$. Let $e \in C$ be a node. Recall that by Kato's classification, $\overline{\mathcal{M}}_{C,e} \cong Q \oplus_{\mathbb{N}} \mathbb{N}^2$ for some $\mathbb{N} \to Q, 1 \mapsto q_e \neq 0$. Let η_1, η_2 be generic points of the components of C adjacent to e with $\overline{\mathcal{M}}_{C,\eta_i} \cong Q$. We have the following diagram,



where the maps χ_i are induced by generizations $e \to \eta_i$. The diagram defines a map $u_e : \overline{\mathcal{M}}_{X,f(e)} \to \mathbb{Z}$ by the property,

$$f_{\eta_2} \circ \chi_2 - f_{\eta_1} \circ \chi_1 = u_e q_e$$

If u_e is nonzero, there is a unique primitive $\tilde{u}_e \in Hom(\overline{\mathcal{M}}^{gp}_{X,f(e)}, \mathbb{Z})$ and $w_e > 0$ such that $u_e = w_e \tilde{u}_e$. The number w_e is called *the weight* of e. We define the monoid,

$$Q^{\vee}_{basic} := \left\{ ((V_\eta)_\eta, (l_e)_e) \in \bigoplus_\eta P^{\vee}_\eta \oplus \bigoplus_e \mathbb{N} \,\middle|\, V_{\eta_2} \circ \chi_2 - V_{\eta_1} \circ \chi_1 = l_e u_e \text{ for all } e \right\}$$

where the first sum runs over generic points η of irreducible components of C and the second sum runs over nodes e. The monoid $Q^\vee_{basic}$ is called the *basic monoid*.

When W is the standard log point $pt_{\mathbb{N}}$, there is link between stable log maps and tropical geometry. One can associate the dual of the basic monoid to a moduli of tropical curves ([**KLR**], Section 4).

There is a well defined structure map $Q \to Q^\vee_{basic}$ (see [**GS13**], Definition 3.2).

Definition 22. *A stable log map $f : C/W \to X/S$ is called basic if the structure map $Q \to Q^\vee_{basic}$ is an isomorphism.*

By [**GS13**], Proposition 1.24, any stable log map arises from the pullback of basic stable log map with the same underlying ordinary stable map.

Denote $\overline{\mathcal{M}}_{g,n}(X/S,\beta)$ to be the moduli space of genus g, n-marked, basic stable log maps to X in class β. We will sometimes abbreviate this space as $\overline{\mathcal{M}}(X)$.

Theorem 11. *If β is combinatorially finite (see* [**GS13**]*, Definition 3.3), then $\overline{\mathcal{M}}_{g,n}(X/S,\beta)$ is a proper Deligne-Mumford stack of finite type over S.*

The moduli space of curves naturally admits a divisoral log structure given by its normal crossings boundary. We denote $\mathfrak{M}_{g,n}$ to be the Artin stack of genus g, n-marked, pre-stable log curves. We will sometimes abbreviate it as $\mathfrak{M}$.

Using Olsson's results on log cotangent complexes [**Ol**], the moduli space of basic stable log maps $\overline{\mathcal{M}}(X)$ carries a perfect obstruction theory in the sense of [**BF**] over the log stack of pre-stable curves $\mathfrak{M}$. It is given by,

$$E^\bullet = (R^\bullet \pi_* f^* \Theta_{X/S})^\vee \to L^\bullet_{\overline{\mathcal{M}}(X)/\mathfrak{M}}$$

where $L^{\bullet}_{\overline{\mathcal{M}}(X)/\mathfrak{M}}$ is the relative cotangent complex of the forgetful morphism $\rho : \overline{\mathcal{M}}(X) \to \mathfrak{M}$, and π and f are the universal maps for the universal curve of $\overline{\mathcal{M}}(X)$. The construction of the virtual fundamental class $[\overline{\mathcal{M}}(X)]^{vir}$ then follows from the machinery of [**BF**], and one define logarithmic Gromov-Witten invariants in the usual way by integrating incidence conditions over $[\overline{\mathcal{M}}(X)]^{vir}$.

2.6.2.1. Virtual dimension of stable log maps to $X(\log D)$. Now, suppose that X is a log scheme equipped with a divisorial log structure given by a divisor D. Let $\overline{\mathcal{M}}_{g,n+r}(X/S,\beta)_{\overrightarrow{\ell}}$ be the moduli space of genus g, basic stable log maps of class β with $n+r$ marked points, with the first n points having zero contact order with D, and the other r points having prescribed contact orders $\overrightarrow{\ell} = (\ell_1, \ldots, \ell_r) \in \mathbb{Z}^r_{>0}$ with D and satisfying $\beta \cdot D = \sum_i \ell_i$. A generic curve of class β will intersect D at $\beta \cdot D$ many points and at each point with contact order 1. If we prescribe that some points have contact order $\ell_i > 1$, then each additional contact order cuts the virtual dimension down by 1. Hence, the contact points will cut down the virtual dimension by $\sum_i (\ell_i - 1)$. The virtual dimension is given by,

$$\begin{aligned} \text{virdim } \overline{\mathcal{M}}_{g,n+r}(X,\beta)_\ell &= \int_\beta c_1(TX) + (\dim X - 3)(1-g) + n - \sum_i (\ell_i - 1) \\ &= \int_\beta c_1(TX) + (\dim X - 3)(1-g) + n + r - \beta \cdot D \end{aligned}$$

We will at times suppress the notation $\overrightarrow{\ell}$ and just write $\overline{\mathcal{M}}_{g,n+r}(X/S,\beta)_{\overrightarrow{\ell}}$, when the context is clear.

2.6.3. Torically transverse maps

Suppose that X is a toric variety. The moduli space of basic stable log maps $\overline{\mathcal{M}}(X)$ has a open substack of *torically transverse curves.*

Definition 23 ([**NS**]). *A stable log map $f : C \to X$ is torically transverse or tt if no component maps dominantly to a toric divisor, or if it is disjoint from all toric strata of codimension > 2.*

In particular, torically transverse stable maps do not have irreducible components mapping into the toric boundary. We will use the superscript *tt* to denote the open substack of torically transverse curves $\overline{\mathcal{M}}(X)^{tt} \subset \overline{\mathcal{M}}(X)$. If the $\overline{\mathcal{M}}(X)$ is log smooth over $\mathfrak{M}$, then $\overline{\mathcal{M}}(X)^{tt}$ is a dense open subset.

The open substack of torically transverse curves has trivial log structure. The stable log map in which the target X has trivial log structure is isomorphic to the underlying ordinary stable map under the functor the forgets the log structure. Hence, the log Gromov-Witten invariants of trivial logarithmic structure on X coincide with the ordinary Gromov-Witten invariants of X.

We have the following lemma on the intersection number of tt-maps,

Proposition 1 ([**NS**], Lemma 4.2). *Suppose that X is a toric variety with divisorial log structure given by the toric boundary ∂X. If $\varphi : C \to X$ is torically transverse stable log map, then,*

$$\sum_i w_i u_i = 0$$

where $w_i = \deg \varphi^* D_i$ are the intersection number of $\varphi(C)$ with each divisor D_i, and u_i are primitive generators of the rays of the fan of X.

Remark 4 (Log GW-invariants vs. relative GW-invariants)**.** *The central issue in relative Gromov-Witten theory is that the limit of curves intersecting a divisor properly may not have proper intersection, as the limit could sink into the divisor. An expanded degeneration of the target is a finite number of $\mathbb{P}^1$-bundles or "bubbles" glued to the divisor in order to attain proper intersection. Jun Li shows that stable maps into the moduli of expanded degenerations is proper and hence has a virtual fundamental class* [**Li**]*. When the divisor is smooth, the log invariants are equivalent to relative invariants.*

2.6.4. Birational invariance of log GW theory

We mention the main theorem of [**AW**], which gives a birational condition for when two log schemes have equivalent log Gromov-Witten invariants.

Definition 24. *A log modification is a proper, birational, and logarithmically étale morphism $X \to Y$.*

Example 15. *A toric blow up of toric varieties with the toric log structure is a log modification.*

Theorem 12 ([**AW**])**.** *Given a logarithmic modification $h : X \to Y$ inducing a projection $\pi : \overline{\mathcal{M}}(X) \to \overline{\mathcal{M}}(Y)$, we have,*

$$\pi_*[\overline{\mathcal{M}}(X)]^{vir} = [\overline{\mathcal{M}}(Y)]^{vir}$$

CHAPTER 3

Scattering and Gross-Siebert mirror symmetry for $\mathbb{P}^2$: a primer

3.1. Scattering

Scattering diagrams encode wall crossing structures accounting for certain instanton corrections. Wall crossing has appeared in many contexts including counts of holomorphic discs in a Lagrangian fibration, quiver Donaldson-Thomas invariants, Gromov-Witten theory of blow ups of toric surfaces, counting of geodesics of quadratic differential on a curve, or $N = 2, d = 4$ supersymmetric gauge theory. They were used by Kontsevich-Soibelman to construct non-Archimedean K3 surfaces. In many cases, various enumerative invariants have been shown to satisfy/follow the Kontsevich-Soibelman wall crossing formula (WCF) [**KS**]. In Gross-Siebert mirror symmetry, chambers of the scattering diagram describe charts of the mirror toric degeneration. Given a nilpotent Lie algebra, one can create a wall structure labelled with elements in the associated Lie group. We will not define scattering in this generality, and we refer to [**Man2**] for definitions. Instead, we will explain scattering in the (quantum) tropical vertex group [**GPS**].

3.1.1. Related work

Scattering diagrams can also be expressed in terms of multiplicities of tropical curves. There is a 1-1 correspondence between walls and rational tropical curves, where a wall corresponds to an edge of a tropical curve (see [**GPS**], Theorems 2.4 and 2.8 or [**Gro**], Definition 5.26).

Scattering diagrams can be defined more generally over by taking wall functions to lie in a Lie group corresponding to a nilpotent Lie algebra. Mandel showed that walls of scattering diagrams defined over various nilpotent Lie algebras can be expressed in terms of tropical curves [**Man2**].

In [**GPS**], commutator formulas in the tropical vertex group are shown to be equivalent to calculations of genus 0, relative Gromov-Witten invariants of toric surfaces. In [**Rei**], the same commutator formulas are shown to be equivalent to computing Euler characteristics of moduli of framed quiver representations.

From a 2-cyclic quiver Q, Bridgeland [**Bri**] defines *Hall algebra scattering diagrams* that lie in the space of stability conditions of Q: each indecomposable representation of Q defines a wall in a space of semi-slope stability conditions of Q. The scattering diagrams in Examples 16 and 17 correspond to the scattering diagrams defined by the A_2 quiver and the Kronecker quiver, respectively. In [**CM**], it is shown the Hall algebra broken lines do not satisfy the consistency lemma of [**CPS**].

In mirror symmetry for log Calabi-Yau surfaces with maximal boundary, the results of [**GPS**] are used to prove consistency of the *canonical scattering diagram*, which is defined by the maximal tangency log Gromov-Witten theory of a log Calabi-Yau surface [**GHKK**].

In context of cluster varieties, Fock and Goncharov conjectured that the ring of regular functions of the $\mathcal{X}$ cluster variety is parametrized by integral points of the tropicalization of the $\mathcal{A}$ cluster variety [**FG1**], [**FG2**]. Gross, Hacking, Keel, and Kontsevich proved the conjecture by constructing certain scattering diagrams from cluster algebras [**GHKK**].

In symplectic geometry, Bardwell-Evans, Cheung, Hong, and Lin constructed scattering diagrams from Lagrangian Floer theory of special Lagrangian fibres [**BCHL**], and show that their construction agrees with the scattering diagrams of [**GPS**], [**GHK**].

It is conjectured in [**HSZ**] that certain Baxter operators associated to the Skein algebra of a torus satisfy the Pentagon relation and is proven in [**Hu**]. It invites the possibility for enumerative interpretations of scattering with skein algebras.

3.1.2. The Tropical Vertex group

The tropical vertex group was introduced by Kontsevich and Soibelman in describing wall crossing of Donaldson-Thomas invariants [**KS**]. Elements of the group are formal, one parameter symplectomorphisms of the algebraic torus $T \cong \operatorname{Spec} \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with respect to the holomorphic symplectic form $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$. In this section, we recall some facts about the group. We first give a definition of the tropical vertex group as the exponential of a Lie algebra.

Let $M \cong \mathbb{Z}^2$ be a lattice, and $N = Hom_{\mathbb{Z}}(M, \mathbb{Z})$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. We will write $n \cdot m$ or $m \cdot n$ for the natural evaluation of $n \in N$ on $m \in M$. Let R be an Artin local $\mathbb{C}$-algebra, and $\mathfrak{m}_R \subseteq R$ be the maximal ideal. One may take R to be $\mathbb{C}[t]/(t^k)$ for any k.

For a $\mathbb{C}$-algebra A, we define $A \hat{\otimes}_{\mathbb{C}} R \coloneqq \lim_{\leftarrow} A \otimes_{\mathbb{C}} R/\mathfrak{m}_R^k$. For this section, we let $A = \mathbb{C}[M]$. Define the module of log derivations to be,

$$\Theta(\mathbb{C}[M] \hat{\otimes}_{\mathbb{C}} R) = Hom(M, \mathbb{C}[M] \hat{\otimes}_{\mathbb{C}} R) = (\mathbb{C}[M] \hat{\otimes}_{\mathbb{C}} R) \otimes_{\mathbb{Z}} N$$

A log derivation ξ induces an ordinary derivation ξ' of $\mathbb{C}[M]\hat{\otimes}_{\mathbb{C}}R$ over R by,

$$\xi'(z^m) = \xi(m)z^m$$

We write an element $a \otimes n \in (\mathbb{C}[M]\hat{\otimes}_{\mathbb{C}}R) \otimes_{\mathbb{Z}} N$ as $a\partial_n$. It is an ordinary derivation on $\mathbb{C}[M]$ defined as,

$$(a\partial_n)(z^m) = a(m \cdot n)z^m$$

Define $\mathfrak{g}_R := \mathfrak{m}_R\Theta(\mathbb{C}[M]\hat{\otimes}_{\mathbb{C}}R)$. It is naturally a Lie algebra with bracket defined by,

$$[z^m\partial_n, z^{m'}\partial_{n'}] = (z^m\partial_n(z^{m'}))\partial_{n'} - (z^{m'}\partial_{n'}(z^m))\partial_n$$

Given $\xi \in \mathfrak{g}_R$, we obtain an element $\exp(\xi) \in Aut_R(\mathbb{C}[M]\hat{\otimes}_{\mathbb{C}}R)$ defined as,

$$\exp(\xi)(a) = Id(a) + \sum_{i=1}^{\infty} \frac{(\xi')^i(a)}{i!}$$

which is well-defined since R is complete.

Let $\mathfrak{h}_R \subseteq \mathfrak{g}_R$ be defined as,

$$\mathfrak{h}_R := \bigoplus_{m \in M \setminus \{0\}} \mathbb{C}z^m(\mathfrak{m}_R \otimes m^{\perp}) \tag{3.1}$$

which is closed under the Lie bracket for $\mathfrak{g}_R$. The *tropical vertex group* $\mathbb{V}_R$ is defined as,

$$\mathbb{V}_R := \{\exp(\xi) | \xi \in \mathfrak{h}_R\}$$

Remark 5. *The parameter t encodes consistency of the scattering diagram and subsequently the construction of the mirror toric degeneration. The exponent of t^k is given by the image of a monomial z^m of a piecewise linear function φ.*

3.1.3. Scattering diagrams

In this section, R is again a local, Artinian $\mathbb{C}$-algebra, and $\mathbb{C}[M]$ is the algebra of the torus given by Laurent polynomials.

Definition 25. *A ray or wall is a pair (σ, f_σ) such that,*

(1) $\sigma \subseteq M_{\mathbb{R}}$ and $\sigma = x_\sigma + \mathbb{R}_{\geq 0} m_\sigma$ for some $x_\sigma \in M_{\mathbb{R}}$ and $m_\sigma \in M$ is a primitive direction vector of the ray.

(2) $f_\sigma \in \mathbb{C}[z^{\pm m_\sigma}]\hat{\otimes}_{\mathbb{C}} R$.

(3) $f_\sigma \equiv 1 \mod z^{\pm m_\sigma}\mathfrak{m}_R$

We call f_σ the wall function or wall automorphism of σ. The choice of sign for $z^{\pm m_\sigma}$ is determined by whether the ray is incoming or outgoing. We follow the convention that $+$ means incoming and $-$ means outgoing.

Definition 26. *A scattering diagram $\mathfrak{D}$ is a set of rays such that for every power $k > 0$, there are only finitely many rays $(\sigma, f_\sigma) \in \mathfrak{D}$ with $f_\sigma \not\equiv 1 \mod \mathfrak{m}_R^k$.*

Definition 27. *For a scattering diagram $\mathfrak{D}$, we define the support of $\mathfrak{D}$ to be,*

$$Supp(\mathfrak{D}) \coloneqq \bigcup_{\sigma \in \mathfrak{D}} \sigma \subseteq M_{\mathbb{R}}$$

and we say that the joints of $\mathfrak{D}$ are defined as,

$$Sing(\mathfrak{D}) := \bigcup_{\sigma\in\mathfrak{D}} \partial\sigma \cup \bigcup_{\substack{\sigma_1,\sigma_2,\\ \dim\sigma_1\cap\sigma_2=0}} \sigma_1\cap\sigma_2$$

where $\partial\sigma := x_\sigma$. We call the connected components of $M_\mathbb{R} \setminus Supp(\mathfrak{D})$ the chambers of $\mathfrak{D}$.

3.1.4. Consistency

Given a scattering diagram $\mathfrak{D}_l$ consistent to order $\mathfrak{m}_R^l$, suppose that $\gamma : [0,1] \to M_\mathbb{R}$ is an immersed path that transversely crosses walls $\{(\sigma_i, f_{\sigma_i}) | 1 \le i \le n\}$ at time points t_i with $0 < t_1 < \ldots < t_n < 1$. For each wall σ_i, let $n_i \in N$ be the normal vector satisfying $\langle n_i, \gamma'(t_i)\rangle < 0$. Define *the path ordered product* $\Phi_{\gamma,l}$ in $\mathfrak{D}_l$ to be,

$$\Phi_{\gamma,l} = \prod_{i=1}^{n} \exp(\log(f_{\sigma_i})\partial_{n_i})$$

Define $\Phi_\gamma(z^m) := \lim_{l\to\infty} \Phi_{\gamma,l}(z^m)$.

Definition 28. *A scattering diagram $\mathfrak{D}$ is consistent to order k if for any closed loop γ, we have $\theta_\gamma = Id \mod \mathfrak{m}_R^k$. We say it is consistent if $\theta_\gamma = Id$ for any γ.*

We have the following lemma, due to Kontsevich and Soibelman.

Lemma 2 ([**KS**], "Kontsevich-Soibelman consistency lemma")**.** *Let $\mathfrak{D}$ be a scattering diagram. Then there exists a unique consistent scattering diagram $S(\mathfrak{D})$ obtained from $\mathfrak{D}$ by adding only outgoing rays.*

Proof. We prove the statement by induction on $R/\mathfrak{m}_R^k$ for $k \ge 1$, showing that there exists a $\mathfrak{D}_k$ such that,

$$\Phi_{\gamma,\mathfrak{D}_k} \equiv Id \mod \mathfrak{m}_R^k$$

for all closed loops γ. When $k = 1$, we have $f_\sigma \equiv 1 \mod \mathfrak{m}_R$ by definition. Hence, we can take $\mathfrak{D}_1 = \mathfrak{D}$.

Now, suppose that we have constructed $\mathfrak{D}_k$ that is consistent to $R/\mathfrak{m}_R^k$, i.e.

$$\prod_{(\sigma,f_\sigma)\in\mathfrak{D}_{k-1}} \exp(\log(f_{\sigma_i})\partial_{n_i}) \equiv 1 \mod \mathfrak{m}_R^k$$

Let $\mathfrak{D}'_k \subset \mathfrak{D}_k$ be the set of rays σ' such that $f_{\sigma'} \not\equiv 1 \mod \mathfrak{m}_R^{k+1}$. Let $p \in Sing(\mathfrak{D}'_k)$, and γ_p a simple closed loop around p that does not contain any other points of $Sing(\mathfrak{D}'_k)$. Clearly, we have,

$$\Phi_{\gamma_p,\mathfrak{D}_k} \equiv \Phi_{\gamma_p,\mathfrak{D}'_k} \mod \mathfrak{m}_R^{k+1}$$

By the induction hypothesis, we can uniquely write,

$$\Phi_{\gamma_p,\mathfrak{D}'_k} = \exp\left(\sum_{i=1}^{l_p} \log(f_{\sigma_i})\partial_{n_i}\right)$$

with $f_{\sigma_i} \in \mathbb{C}[z^{\pm m_{\sigma_i}}]\hat{\otimes}_{\mathbb{C}} R/\mathfrak{m}_R^k$. Let

$$\mathfrak{D}_p := \{(p + \mathbb{R}_{\geq 0} m_{\sigma_i}, \exp(-\log(f_{\sigma_i})\partial_{n_i}))|1 \leq i \leq l_p\}$$

Since $\mathfrak{m}_R^k \cdot \mathfrak{m}_R^k \subset \mathfrak{m}_R^{2k} \equiv 0 \mod \mathfrak{m}_R^{k+1}$, we have that,

$$\Phi_{\gamma_p,\mathfrak{D}'_k \cup \mathfrak{D}_p} \equiv 1 \mod \mathfrak{m}_R^{k+1}$$

Let $\mathfrak{D}_{k+1} := \mathfrak{D}_k \bigcup_p \mathfrak{D}_p$, i.e. we add the necessary rays for consistency back to $\mathfrak{D}_p$. We see that $\mathfrak{D}_{k+1}$ is consistent modulo $\mathfrak{m}_R^{k+1}$. Thus, we take $S(\mathfrak{D})$ to be the union of all $\mathfrak{D}_k$ for all k. It may have infinitely many rays. □

We will write the consistent completion $S(\mathfrak{D})$ also as D_∞.

Next, we highlight that cluster transformations are specific instances of elements in the tropical vertex.

3.1.5. Poisson torus algebra

Recall that $M \cong \mathbb{Z}^2$ is a lattice, and consider the group algebra $\mathbb{C}[M]$. Elements in $\mathbb{C}[M]$ are finite $\mathbb{C}$-linear combinations of formal variables or *monomials* z^m for $m \in M$, i.e.

$$\mathbb{C}[M] = \bigoplus_{m \in M} \mathbb{C} z^m$$

The multiplication is defined as $z^m \cdot z^{m'} = z^{m+m'}$. Choose an orientation on M, i.e. a skew-symmetric, bilinear form $\langle , \rangle$ so that $\bigwedge^2 M \cong \mathbb{Z}$. Let $\{m_1, m_2\} \subset M$ be a $\mathbb{Z}$-basis with $\langle m_1, m_2 \rangle = 1$. Then, we may write $\mathbb{C}[M] = \mathbb{C}[z^{\pm m_1}, z^{\pm m_2}]$, and hence $\operatorname{Spec} \mathbb{C}[M] \cong (\mathbb{C}^*)^2$. We denote T to be the algebraic torus $(\mathbb{C}^*)^2$.

Characters of T form a lattice $Hom_{\mathbb{Z}}(T, \mathbb{C}^*)$ and are spanned by elements of the form,

$$(x, y) \mapsto x^a y^b$$

for $m := (a, b) \in \mathbb{Z}^2$. We denote the character as z^m for $z = (x, y) \in (\mathbb{C}^*)^2$. We see that characters form a basis of the algebra of functions $\Gamma(\mathcal{O}_T)$ of T, or $\mathbb{C}[M]$ as above.

We define the bracket on $\Gamma(\mathcal{O}_T)$ by,

$$\{z^m, z^{m'}\} := \langle m, m'\rangle z^{m+m'}$$

which makes $\Gamma(\mathcal{O}_T)$ a Poisson algebra. It has a corresponding algebraic symplectic form $\Omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$.

3.1.6. Cluster transformations are in the Tropical Vertex

Now, let $R = \mathbb{C}[[t]]$, and consider the tropical vertex $V_{\mathbb{R}}$. For any vector $(a,b) \in \mathbb{Z}^2$, we have an automorphism $T_{(a,b),f} \in \mathbb{V}_R$ given explicitly by,

$$T_{(a,b),f}(x) = f^{-b}x, \quad T_{(a,b),f}(y) = f^a x$$

where the function f is of the form $f = 1 + tx^a y^b g(x^a y^b, t) \in \mathbb{C}[M]\hat{\otimes}_{\mathbb{C}}\mathbb{C}[[t]]$ with $g(z,t) \in \mathbb{C}[z][[t]]$.

We notice that cluster transformations of the torus algebra $\Gamma(\mathcal{O}_T)$ are specific examples of elements in the tropical vertex group, and indeed $\Gamma(\mathcal{O}_T) \subset \mathfrak{h}_R$ as Lie algebras: suppose we have a wall σ equipped with the wall function,

$$f_\sigma = 1 + ct^k z^{-m_\sigma}$$

with $c \in \mathbb{C}$ and $k \in \mathbb{N}$. The wall crossing at σ is given by,

$$\exp(\log(f_\sigma)\partial_{n_\sigma}) \cdot z^m = z^m f_\sigma^{n_\sigma \cdot m}$$

which is indeed the cluster transformation,

$$z^m \to z^m f_\sigma^{n_\sigma \cdot m} \tag{3.2}$$

3.1.7. Examples of scattering

Example 16. *Suppose that we have two incoming walls* $(\sigma_1, f_{\sigma_1}) = (\mathbb{R}_{\geq 0} m_1, 1 + z^{m_1})$ *and* $(\sigma_2, f_{\sigma_2}) = (\mathbb{R}_{\geq 0} m_2, 1 + z^{m_2})$ *with* $\langle m_1, m_2 \rangle = 1$ *that meet at the origin. We complete this to a consistent diagram by adding 3 outgoing rays, namely* $(\sigma_3, f_{\sigma_3}) = (\mathbb{R}_{\geq 0}(-m_1), 1 + z^{m_1})$, $(\sigma_4, f_{\sigma_4}) = (\mathbb{R}_{\geq 0}(-m_2), 1 + z^{m_2})$ *and* $(\sigma_5, f_{\sigma_5}) = (\mathbb{R}_{\geq 0}(-m_1 - m_2), 1 + z^{m_1+m_2})$*. One can check by hand that,*

$$f_{\sigma_1} f_{\sigma_2}(z^m) = f_{\sigma_4} f_{\sigma_5} f_{\sigma_3}(z^m)$$

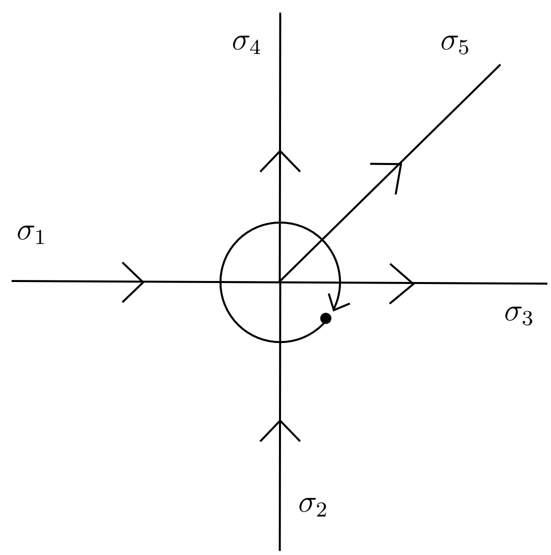


Figure 3.1. Two ingoing walls $\sigma_1 = (\mathbb{R}_{\geq 0}(-1,0), 1 + z^{(-1,0)})$ and $\sigma_2 = (\mathbb{R}_{\geq 0}(0,-1), 1 + z^{(0,-1)})$. Consistency is obtained by adding three outgoing walls $\sigma_3 = (\mathbb{R}_{\geq 0}(1,0), 1 + z^{(-1,0)})$, $\sigma_4 = (\mathbb{R}_{\geq 0}(0,1), 1 + z^{(0,-1)})$, and $\sigma_5 = (\mathbb{R}_{\geq 0}(1,1), 1 + z^{(-1,-1)})$.

Example 17 ([**BBvG**]). *Suppose that we have two incoming walls* $(\sigma_1, f_{\sigma_1}) = (\mathbb{R}_{\geq 0} m_1, 1 + z^{m_1})$ *and* $(\sigma_2, f_{\sigma_2}) = (\mathbb{R}_{\geq 0} m_2, 1 + z^{m_2})$ *with* $\langle m_1, m_2 \rangle = 2$ *that meet at the origin. For simplicity, let's assume* $m_1 = (-1, 0)$ *and* $m_2 = (0, -2)$. *Consistency is obtained by adding the following 3 families of outgoing rays,*

(1) $(\sigma_{nm_1+(n+1)m_2}, f_{nm_1+(n+1)m_2}) \coloneqq (\mathbb{R}_{\geq 0}(-nm_1 - (n+1)m_2), 1 + z^{nm_1+(n+1)m_2})$, *for* $n \geq 0$

(2) $(\sigma_{(n+1)m_1+nm_2}, f_{(n+1)m_1+nm_2}) \coloneqq (\mathbb{R}_{\geq 0}(-(n+1)m_1 - nm_2), 1 + z^{(n+1)m_1+nm_2})$, *for* $n \geq 0$

(3) A single ray $(\sigma_{m_1+m_2}, f_{m_1+m_2}) \coloneqq (\mathbb{R}_{\geq 0}(-m_1 - m_2), (1 - z^{m_1+m_2})^{-2})$

Consistency of the diagram implies that,

$$f_{(-1,0)} f_{(0,-2)} = f_{(0,-2)} f_{(1,-4)} f_{(2,-6)} \cdots f^{-2}_{(-1,-2)} \cdots f_{(-3,-4)} f_{(-2,-2)} f_{(-1,0)}$$

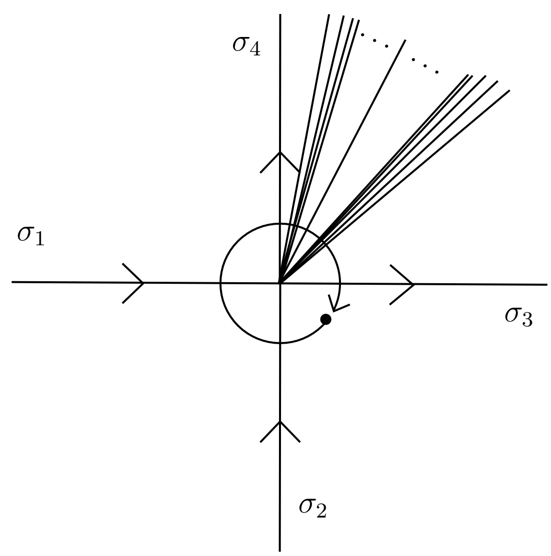


Figure 3.2. Scattering of two ingoing rays with directions m_1 and m_2 satisfying $\langle m_1, m_2 \rangle = 2$, with an infinite number of outgoing rays in the consistent diagram.

Remark 6. *When* $\langle m_1, m_2 \rangle = 3$*, the resulting outgoing walls from consistency do not seem to have an explicit description, however there is a region in which the walls are dense.*

3.2. Quantum scattering

Cluster varieties admit quantizations by suitably quantizing the Poisson torus algebra. Quantum cluster transformations are given by conjugation by the quantum dilogarithm. As in the classical case, quantum cluster transformations are elements of the quantum tropical vertex group of Kontsevich-Soibelman. We quantize the scattering diagram obtained from the Gross-Siebert program by taking wall functions to be automorphisms of the quantum torus algebra. We show that quantizing maintains consistency of the scattering diagram.

3.2.1. Related work

Quantum scattering has appeared in various work connecting higher genus Gromov-Witten theory with tropical geometry. Filipini and Stoppa first showed that qauntum wall crossing is related to Block-Gottsche multiplicity of tropical curves [**FS**]. Bousseau quantizes the canonical scattering diagram of [**GHK**] and consequently constructs a quantization of the GHK mirror family to a log Calabi-Yau surface [**Bou4**]. In [**Bou6**], it is shown that consistency of scattering diagrams in the quantum tropical vertex can be phrased in terms of higher genus log Gromov-Witten invariants with λ_g-insertion of toric surfaces, analogous to the classical result of [**GPS**]. By studying the quantized scattering diagram of an affine cubic surface, Bousseau also shows that the resulting algebra of quantum broken lines is isomorphic to the skein algebra of a 4-punctured sphere, and proves positivity of the bracelets basis [**Bou5**].

3.2.2. Quantum torus algebra

Recall that we have the lattice $M \cong \mathbb{Z}^2$, and a bilinear form $\langle \cdot, \cdot \rangle$ from choosing an orientation on M so that $\bigwedge^2 M \cong \mathbb{Z}$. Let $\mathbf{q} = e^{i\hbar}$ be a formal variable. By the *quantum torus algebra* $\hat{\Gamma}(\mathcal{O}_T)$, we mean the following algebra,

$$\hat{\Gamma}(\mathcal{O}_T) := \mathbb{C}[\mathbf{q}^{\pm\frac{1}{2}}][M] = \bigoplus_{m \in M} \mathbb{C}[\mathbf{q}^{\pm\frac{1}{2}}]z^m$$

It recovers the Poisson torus algebra $\Gamma(\mathcal{O}_T)$ when $q = 1$. We write elements of $\hat{\Gamma}(\mathcal{O}_T)$ as $\hat{z}^m$ for $m \in M$. The noncommutative multiplication is defined as,

$$\hat{z}^m \cdot \hat{z}^{m'} = \mathbf{q}^{\frac{\langle m, m' \rangle}{2}} \hat{z}^{m+m'}$$

which is equivalent to the rule $\hat{z}^m \cdot \hat{z}^{m'} = \mathbf{q}^{\langle m, m' \rangle} \hat{z}^{m'} \cdot \hat{z}^m$. The variable $\mathbf{q}$ is in the center of the algebra. When $\langle \cdot, \cdot \rangle$ is the usual determinant of matrix formed from the two vectors, notice that we have the relation $\hat{z}^{(1,0)} \hat{z}^{(0,1)} = \mathbf{q} \hat{z}^{(0,1)} \hat{z}^{(1,0)}$ or $xy = \mathbf{q}yx$.

3.2.3. Quantum scattering diagrams

The tropical vertex has a quantum version, which we call the *quantum tropical vertex*, denoted by $\mathbb{V}_R^q$. It first appeared in Kontsevich-Soibelman, and used by Bousseau to generalize [**GPS**]. The Lie algebra corresponding to $\mathbb{V}_R^q$ is given by,

$$\mathfrak{h}_R^q := \bigoplus_{m \in M \smallsetminus \{0\}} \mathbb{C}[\mathbf{q}^{\frac{\pm 1}{2}}]\hat{z}^m(\mathfrak{m}_R \otimes m^\perp)$$

(see Equation 3.1 for comparison). Then $\mathbb{V}_R^q := \exp(\mathfrak{h}_R^q)$. Quantum cluster transformations are also elements of $\mathbb{V}_R^q$.

We consider scattering in the quantum tropical vertex. The definitions in classical scattering can be quantized. For simplicity, we will illustrate quantization of cluster transformations.

Definition 29. *Replacing z by $\hat{z}$ in Definitions 25 and 26 for scattering diagrams, we obtain a quantum scattering diagram $\hat{\mathfrak{D}}$. The Kontsevich-Soibleman consistency theorem also applies to quantum scattering diagrams.*

3.2.4. Quantum wall functions

Recall that R is a local Artinian $\mathbb{C}$-algebra. Similar to the classical case, the data of a *quantum wall* $(\sigma, \hat{f}_\sigma)$ consists of a ray σ and a quantized wall function $\hat{f}_\sigma$. Recall that $f_\sigma \in \mathbb{C}[z^{\pm m_\sigma}]\hat{\otimes}_{\mathbb{C}} R$ and satisfies $f_\sigma \equiv 1 \mod z^{\pm m_\sigma}\mathfrak{m}_R$. We write f_σ (with σ an outgoing ray) in the form,

$$f_\sigma = \sum_{k \geq 0} c_k z^{-k m_\sigma}, c_k \in R$$

Definition 30. *The quantization of the wall function f_σ is defined to be,*

$$\hat{f}_\sigma := \sum_{k \geq 0} c_k \mathbf{q}^{\frac{-k}{2}} \hat{z}^{-k m_\sigma} \in \mathbb{C}[\mathbf{q}^{\frac{\pm 1}{2}}][z^{\pm m_\sigma}]\hat{\otimes}_{\mathbb{C}} R.$$

It satisfies $\lim_{q\to 1}\hat{f}_\sigma = f_\sigma$*, and* $\hat{f}_\sigma \equiv 1 \mod \mathbf{q}^{\frac{\pm 1}{2}} z^{\pm m_\sigma}\mathfrak{m}_R$*. For* $j \in \mathbb{Z}$*, we define the following functions,*

$$\hat{f}_{\sigma,j} \coloneqq \sum_{k\geq 0} c_k \mathbf{q}^{-j-\frac{k}{2}} \hat{z}^{-km_\sigma} \in \mathbb{C}[\mathbf{q}^{\frac{\pm 1}{2}}][z^{\pm m_\sigma}]\hat{\otimes}_{\mathbb{C}} R.$$

Notice that $\hat{f}_\sigma = \hat{f}_{\sigma,0}$.

Example 18. *Suppose we have an outgoing ray* $\sigma = \mathbb{R}_{\geq 0}(-1,3)$ *with wall function,*

$$f_\sigma = 1 + t^3 z^{(1,-3)}$$

Then, its quantized wall functions are,

$$\hat{f}_\sigma = 1 + \mathbf{q}^{\frac{-1}{2}} t^3 \hat{z}^{(1,-3)}, \quad \hat{f}_{\sigma,j} = 1 + \mathbf{q}^{-j-\frac{1}{2}} t^3 \hat{z}^{(1,-3)}, \quad j \in \mathbb{Z}$$

Definition 31. *Suppose we have a quantum wall function* $\hat{f}_\sigma$ *satisfying* $\hat{f}_\sigma \equiv 1 \mod \mathbf{q}^{\frac{\pm 1}{2}} z^{\pm m_\sigma}\mathfrak{m}_R$*. We write*

$$\log \hat{f}_\sigma = \sum_{k\geq 1} d_k \hat{z}^{\pm k m_\sigma}, d_k \in \mathbb{C}[\mathbf{q}^{\frac{\pm 1}{2}}]\hat{\otimes}_{\mathbb{C}} R$$

Define the Hamiltonian $\hat{H}_\sigma$ *associated to* $\hat{f}_\sigma$ *by,*

$$\hat{H}_\sigma \coloneqq \sum_{k\geq 1} \frac{d_k}{\mathbf{q}^{-k} - 1} \hat{z}^{\pm k m_\sigma}$$

Example 19. *Let* $\hat{f}_\sigma = 1 + \mathbf{q}^{\frac{-1}{2}} t^3 \hat{z}^{(1,-3)}$*. We have*

$$\log \hat{f}_\sigma = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} \mathbf{q}^{\frac{-k}{2}} t^{3k} \hat{z}^{k(1,-3)}$$
$$= -\sum_{k\geq 1} \frac{(-1)^k}{k} \frac{\mathbf{q}^{-k}-1}{\mathbf{q}^{\frac{k}{2}} - \mathbf{q}^{\frac{-k}{2}}} t^{3k} \hat{z}^{k(1,-3)}$$

The Hamiltonian associated to $\hat{f}_\sigma$ is,

$$\hat{H}_\sigma = -\sum_{k\geq 1} \frac{(-1)^k}{k} \frac{t^{3k}\hat{z}^{k(1,-3)}}{\mathbf{q}^{\frac{k}{2}} - \mathbf{q}^{\frac{-k}{2}}}$$

3.2.5. Quantum cluster transformations

The classical dilogarithm function can be used to express volumes of hyperbolic manifolds. We refer to [**FK**] for facts about the quantum dilogarithm.

Definition 32. *The quantum dilogarithm $\Psi_q(\hat{z}^m)$ is the following function on the quantum torus algebra $\hat{\Gamma}(\mathcal{O}_T)$,*

$$\Psi_q(\hat{z}^m) := \exp\left(-\sum_{k\geq 1} \frac{1}{k} \frac{\hat{z}^{km}}{\mathbf{q}^{\frac{k}{2}} - \mathbf{q}^{\frac{-k}{2}}}\right) = \prod_{k\geq 0} \frac{1}{1-\mathbf{q}^{k+\frac{1}{2}}\hat{z}^m}$$

We see that the Hamiltonian associated to Ψ_q is $-\sum_{k\geq 1} \frac{1}{k} \frac{\hat{z}^{km}}{\mathbf{q}^{\frac{k}{2}} - \mathbf{q}^{\frac{-k}{2}}}$.

The quantum dilogarithm satisfies $\Psi_{q^{-1}} = \Psi_q^{-1}$, and hence $Ad^{-1}_{\Psi_q(\hat{z}^m)} = Ad_{\Psi_{q^{-1}}(\hat{z}^m)}$. Conjugation by the quantum dilogarithm or $Ad_{\Psi_q(\hat{z}^m)}$ is an automorphism of the quantum torus algebra. As $\mathbf{q} \to 1$, $Ad_{\Psi_q(\hat{z}^m)}$ becomes the classical cluster transformation in Equation 3.2.

3.2.6. Quantum wall automorphism

Here we explain the quantum wall automorphism for the quantum tropical vertex group $\mathbb{V}_R^q$. Quantum cluster transformations are specific elements in $\mathbb{V}_R^q$.

Definition 33. *Let $(\sigma, \hat{f}_\sigma)$ be a quantum wall with Hamiltonian $\hat{H}_\sigma$. The quantum wall automorphism associated to $(\sigma, \hat{f}_\sigma)$ is $Ad_{\exp(\hat{H}_\sigma)} \in \mathbb{V}_R^q$.*

Lemma 3. *The quantum wall automorphism $Ad_{\exp(\hat{H}_\sigma)}$ can be rewritten as,*

$$\hat{z}^m \to \begin{cases} \hat{z}^m \prod_{j=0}^{\langle \pm m_\sigma, m\rangle - 1} \hat{f}_{\sigma,j}, \text{ for } \langle \pm m_\sigma, m\rangle \geq 0 \\ \\ \hat{z}^m \prod_{j=0}^{|\langle m_\sigma, m\rangle| - 1} \hat{f}_{\sigma,-j-1}^{-1}, \text{ for } \langle \pm m_\sigma, m\rangle < 0 \end{cases}$$

with the $\hat{f}_{\sigma,j}$ defined in Definition 30.

The inverse quantum wall automorphism $Ad^{-1}_{\exp(\hat{H}_\sigma)} = Ad_{\exp(-\hat{H}_\sigma)}$ can be rewritten as,

$$\hat{z}^m \to \begin{cases} \hat{z}^m \prod_{j=0}^{\langle \pm m_\sigma, m\rangle - 1} \hat{f}_{\sigma,j}^{-1}, \text{ if } \langle \pm m_\sigma, m\rangle \geq 0 \\ \\ \hat{z}^m \prod_{j=0}^{|\langle m_\sigma, m\rangle| - 1} \hat{f}_{\sigma,-j-1}, \text{ if } \langle \pm m_\sigma, m\rangle < 0 \end{cases}$$

A universal choice of sign is made throughout if σ is ingoing $(+)$ or outgoing $(-)$.

Proof. This is described in [**Bou3**], Lemma 3.3. We give here the expression for the second map. Write,

$$\hat{H}_\sigma = \sum_{k\geq 1} H_k \hat{z}^{-km_\sigma}, H_k \in \mathbb{C}[q^{\frac{\pm 1}{2}}]\hat{\otimes}_{\mathbb{C}} R$$

Suppose $\langle m_\sigma, m\rangle < 0$. Then we have,

$$\begin{aligned}
Ad^{-1}_{\exp(\hat{H}_\sigma)}(\hat{z}^m) = Ad_{\exp(-\hat{H}_\sigma)}(\hat{z}^m) &= \exp(-\hat{H}_\sigma)\hat{z}^m \exp(\hat{H}_\sigma) \\
&= \exp(-\sum_{k\geq 1} H_k \hat{z}^{-km_\sigma})\hat{z}^m \exp(\sum_{k\geq 1} H_k \hat{z}^{-km_\sigma}) \\
&= \hat{z}^m \exp(\sum_{k\geq 1}(1-\mathbf{q}^{\langle -km_\sigma, m\rangle})H_k \hat{z}^{-km_\sigma}) \\
&= \hat{z}^m \exp(\sum_{k\geq 1}\frac{(1-\mathbf{q}^{\langle -km_\sigma, m\rangle})}{1-\mathbf{q}^k}(1-\mathbf{q}^k)H_k \hat{z}^{-km_\sigma}) \\
&= \hat{z}^m \exp(-\sum_{k\geq 1}\sum_{j=0}^{|\langle m_\sigma, m\rangle|-1}\mathbf{q}^{kj}(\mathbf{q}^k-1)H_k \hat{z}^{-km_\sigma}) \\
&= \hat{z}^m \prod_{j=0}^{|\langle m_\sigma, m\rangle|-1}\hat{f}^{-1}_{\sigma,-j-1}
\end{aligned}$$

The case when $\langle m_\sigma, m\rangle \geq 0$ is similar. □

Remark 7. *When writing* $\exp(\hat{H}_\sigma)$*, we strictly mean the derivation given by the Poisson bracket with* $\hat{H}$*. We refer to* [**Bou2**]*, Section 1.3 for more details.*

Example 20. *Let* $(\sigma, \hat{f}_\sigma)$ *be the quantum wall in Example 19 with* $m_\sigma = (-1,3)$*. The Hamiltonian associated to* $\hat{f}_\sigma$ *is,*

$$\hat{H}_\sigma = -\sum_{k\geq 1}\frac{(-1)^k}{k}\frac{t^{3k}\hat{z}^{k(1,-3)}}{\mathbf{q}^{\frac{k}{2}}-\mathbf{q}^{\frac{-k}{2}}}$$

By Lemma 3, the quantum wall automorphism associated to $(\sigma, \hat{f}_\sigma)$ is,

$$\hat{z}^m \to \begin{cases} \hat{z}^m(1+\mathbf{q}^{\frac{-1}{2}}t^3\hat{z}^{(1,-3)})(1+\mathbf{q}^{\frac{-3}{2}}t^3\hat{z}^{(1,-3)})\dots(1+\mathbf{q}^{-\langle m_\sigma,m\rangle+\frac{1}{2}}t^3\hat{z}^{(1,-3)}), & \text{if } \langle m_\sigma, m\rangle \geq 0 \\ \\ \hat{z}^m(1+\mathbf{q}^{\frac{1}{2}}t^3\hat{z}^{(1,-3)})^{-1}(1+\mathbf{q}^{\frac{3}{2}}t^3\hat{z}^{(1,-3)})^{-1}\dots(1+\mathbf{q}^{|\langle m_\sigma,m\rangle|-\frac{1}{2}}t^3\hat{z}^{(1,-3)})^{-1}, & \text{if } \langle m_\sigma, m\rangle < 0 \end{cases}$$

with inverse,

$$\hat{z}^m \to \begin{cases} \hat{z}^m(1+\mathbf{q}^{\frac{-1}{2}}t^3\hat{z}^{(1,-3)})^{-1}(1+\mathbf{q}^{\frac{-3}{2}}t^3\hat{z}^{(1,-3)})^{-1}\dots(1+\mathbf{q}^{-\langle m_\sigma,m\rangle+\frac{1}{2}}t^3\hat{z}^{(1,-3)})^{-1}, & \text{if } \langle m_\sigma, m\rangle \geq 0 \\ \\ \hat{z}^m(1+\mathbf{q}^{\frac{1}{2}}t^3\hat{z}^{(1,-3)})(1+\mathbf{q}^{\frac{3}{2}}t^3\hat{z}^{(1,-3)})\dots(1+\mathbf{q}^{|\langle m_\sigma,m\rangle|-\frac{1}{2}}t^3\hat{z}^{(1,-3)}), & \text{if } \langle m_\sigma, m\rangle < 0 \end{cases}$$

3.2.7. Path ordered quantum product

We define here for completeness a path ordered product in the quantum scattering diagram. Let $\gamma : [0,1] \to \mathbb{R}^2$ be an immersed path that transversely crosses walls $\sigma_1, \dots, \sigma_k$ at points $\gamma(t_i)$ with $t_i < t_j$ for $i < j$ in the scattering diagram $\mathfrak{D}_l$ for some $l > 0$. At each wall σ_i, we apply $Ad^{sgn\langle \pm m_{\sigma_i}, \gamma'(t_i)\rangle}_{\Phi_q(\hat{z}^{m_{\sigma_i}})}$, where the $\pm$ is chosen according to whether σ_i is ingoing $(+)$ or outgoing $(-)$. Notice that this rule takes into account whether the wall is ingoing or outgoing and the orientation of the crossing. The *path ordered quantum product* $\hat{\Phi}_\gamma$ associated to γ in $\mathfrak{D}_l$ is defined as,

$$\hat{\Phi}_{\gamma,l}(\hat{z}^m) \coloneqq Ad^{-sgn\langle \pm m_{\sigma_k}, \gamma'(t_k)\rangle}_{\Phi_q(\hat{z}^{\pm m_{\sigma_k}})} \circ \dots \circ Ad^{-sgn\langle \pm m_{\sigma_1}, \gamma'(t_1)\rangle}_{\Phi_q(\hat{z}^{\pm m_{\sigma_1}})}(\hat{z}^m)$$

Define $\hat{\Phi}_\gamma(\hat{z}^m) \coloneqq \lim_{l\to\infty} \hat{\Phi}_{\gamma,l}(\hat{z}^m)$, which is a path ordered product in $S(\mathfrak{D})$.

3.2.8. Examples of quantum scattering

Example 21. *Suppose that we have an ingoing wall* $(\sigma_1, \hat{f}_{\sigma_1}) = (\mathbb{R}_{\geq 0}(-1,0), 1+\mathbf{q}^{\frac{-1}{2}}\hat{z}^{(-1,0)})$ *and an outgoing wall* $(\sigma_2, \hat{f}_{\sigma_2}) = (\mathbb{R}_{\geq 0}(1,0), 1+\mathbf{q}^{\frac{-1}{2}}\hat{z}^{(-1,0)})$ *Suppose that we have a loop around the origin that transversely intersects the walls* σ_1 *and* σ_2 *at times* $t_1 < t_2$*, with* $\gamma'(t_1) = (0,-1)$ *and* $\gamma'(t_2) = (0,1)$*. Then, the automorphism given by the path* γ *is* $\hat{\Phi}_\gamma = Id$ *with* $\hat{\Phi}_\gamma(\hat{z}^m) = Ad^{-1}_{\Psi_q(\hat{z}^{m_\sigma})} Ad_{\Psi_q(\hat{z}^{m}(\hat{f}_\sigma))}(\hat{z}^m) = \hat{z}^m$.

Example 22. *Suppose that we have two incoming rays,*

$$(\sigma_1, \hat{f}_{\sigma_1}) = (\mathbb{R}_{\geq 0}(-1,0), 1+\mathbf{q}^{\frac{-1}{2}}\hat{z}^{(-1,0)})$$

$$(\sigma_2, \hat{f}_{\sigma_2}) = (\mathbb{R}_{\geq 0}(0,-1), 1+\mathbf{q}^{\frac{-1}{2}}\hat{z}^{(0,-1)})$$

and three outgoing rays,

$$(\sigma_3, \hat{f}_{\sigma_3}) = (\mathbb{R}_{\geq 0}(1,0), 1+\mathbf{q}^{\frac{-1}{2}}\hat{z}^{(-1,0)})$$

$$(\sigma_4, \hat{f}_{\sigma_4}) = (\mathbb{R}_{\geq 0}(1,1), 1+\mathbf{q}^{\frac{-1}{2}}\hat{z}^{(-1,-1)})$$

$$(\sigma_5, \hat{f}_{\sigma_5}) = (\mathbb{R}_{\geq 0}(0,1), 1+\mathbf{q}^{\frac{-1}{2}}\hat{z}^{(0,-1)})$$

The diagram in Figure 3.1 is a consistent quantum scattering diagram after replacing σ *with* $\hat{\sigma}$*.*

Let γ_1 *be a path starting in the upper left quadrant and crossing* σ_1 *and then* σ_2 *before ending in the lower right quadrant. Let* γ_2 *be a path starting in the upper left quadrant and crossing* σ_5, σ_4 *and* σ_3*, before ending in the lower quadrant. We check that* $\Phi_{\gamma_1} = \Phi_{\gamma_2}$*.*

By the rule for the quantum path ordered product, we have that,

$$\Phi_{\gamma_1} = Ad_{\Psi_q(\hat{z}^{(0,-1)})} Ad_{\Psi_q(\hat{z}^{(-1,0)})}$$

and

$$\Phi_{\gamma_2} = Ad_{\Psi_q(\hat{z}^{(-1,0)})} Ad_{\Psi_q(\hat{z}^{(-1,-1)})} Ad_{\Psi_q(\hat{z}^{(0,-1)})}$$

Indeed, $\Phi_{\gamma_1} = \Phi_{\gamma_2}$ *by the Pentagon identity for the quantum dilogarithm* [1]. *Thus, the diagram is consistent.*

Remark 8. *The* $\mathbf{q}$*-multiplicity of tropical curves naturally appears in the application of quantum cluster transformations. Recall that the* $\mathbf{q}$*-multiplicity of a vertex* V *of a tropical curve (Definition 11) is defined as,*

$$m_V(q) = \frac{\mathbf{q}^{\frac{m_V}{2}} - \mathbf{q}^{\frac{-m_V}{2}}}{\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}}}$$

Notice these quantities have a $\mathbf{q} \mapsto \mathbf{q}^{-1}$ *symmetry.*

Suppose we have a quantum wall function $\hat{f}_\sigma = 1 + \mathbf{q}^{\frac{-1}{2}} t^k \hat{z}^{-m_\sigma}$ *with* $\langle m_\sigma, m \rangle \geq 0$*. The* $\mathbf{q}$*-wall crossing of* $\hat{z}^m$ *is,*

$$\hat{z}^m \prod_{j=0}^{\langle m_\sigma, m\rangle - 1} \hat{f}_{\sigma,j} = \hat{z}^m (1 + \mathbf{q}^{\frac{-1}{2}} t^k \hat{z}^{-m_\sigma})(1 + \mathbf{q}^{\frac{-3}{2}} t^k \hat{z}^{-m_\sigma}) \dots (1 + \mathbf{q}^{-\langle m_\sigma, m\rangle + \frac{1}{2}} t^k \hat{z}^{-m_\sigma})$$

[1] The Pentagon Identity [**FK**] states that if $\langle m, m' \rangle = 1$, then

$$\Psi_q(\hat{z}^m)\Psi_q(\hat{z}^{m'}) = \Psi_q(\hat{z}^{m'})\Psi_q(\hat{z}^{m+m'})\Psi_q(\hat{z}^m)$$

Consider the t^k-term in the product, which is,

$$\hat{z}^m\Big(\sum_{j=0}^{\langle m_\sigma,m\rangle-1}\mathbf{q}^{\frac{-1}{2}-j}\Big)\hat{z}^{-m_\sigma} = \mathbf{q}^{\frac{\langle m_\sigma,m\rangle}{2}}\Big(\sum_{j=0}^{\langle m_\sigma,m\rangle-1}\mathbf{q}^{\frac{-1}{2}-j}\Big)\hat{z}^{m-m_\sigma}$$
$$= \big(\mathbf{q}^{\frac{\langle m_\sigma,m\rangle-1}{2}}+\ldots+\mathbf{q}^{\frac{-\langle m_\sigma,m\rangle+1}{2}}\big)\hat{z}^{m-m_\sigma}$$
$$= [\langle m_\sigma,m\rangle]_{\mathbf{q}}\hat{z}^{m-m_\sigma}$$

The coefficient $(\mathbf{q}^{\frac{\langle m_\sigma,m\rangle-1}{2}}+\ldots+\mathbf{q}^{\frac{-\langle m_\sigma,m\rangle+1}{2}})$ *is the quantum multiplicity of a vertex with two outgoing edges* m_σ *and* m*. Hence, a quantum broken line can be completed to* $\mathbf{q}$*-refined tropical curve by adding the wall it scatters at.*

3.2.9. Computational schema of quantum wall crossing for $(\mathbb{P}^2, E)$

We explain the computational schema of quantum wall crossing for $(\mathbb{P}^2, E)$. Starting with the quantized Hori-Vafa potential in the central chamber, we wall cross vertically upwards (see Section 3.3 for definitions and Figure 3.5 for the scattering diagram). We refer to Appendix A of [**GRZ**] for a list of the first 18 wall functions in the classical case; by our convention, each of the walls there are outgoing except for walls f_{17} and f_{18}.

The quantization of the first 7 walls for $\mathbb{P}^2$ in the scattering diagram consistent to order t^9 is given by,

- $\hat{f}_1 = \mathbf{q}^{\frac{-1}{2}}t(1+x^{-1}) \quad n_1 = (0,-1)$
- $\hat{f}_2 = 1+\mathbf{q}^{\frac{-1}{2}}t^3xy^{-3} \quad n_2 = (-3,-1)$
- $\hat{f}_3 = 1+\mathbf{q}^{\frac{-1}{2}}t^3x^{-1}y^{-3} \quad n_3 = (3,-1)$
- $\hat{f}_4 = 1+15\mathbf{q}^{\frac{-1}{2}}t^9x^{-1}y^{-9} \quad n_4 = (9,-1)$
- $\hat{f}_5 = 1+\mathbf{q}^{\frac{-1}{2}}t^9x^2y^{-9} \quad n_5 = (-9,-2)$

- $\hat{f}_6 = 1 + \mathbf{q}^{\frac{-1}{2}} t^9 x^{-2} y^{-9} \quad n_6 = (9, -2)$
- $\hat{f}_7 = 1 + 3\mathbf{q}^{\frac{-1}{2}} t^6 x y^{-6} \quad n_7 = (-6, -1)$

Aside from $\hat{f}_1$, we quantized f_σ according to Definition 30. The first wall is special since $\hat{f}_1 \not\equiv 1 \mod (t)$. The quantization $\hat{f}_1$ particularly simplifies subsequent quantum wall crossing, in which resulting coefficients coincide with $\mathbf{q}$-multiplicity of tropical curves.

We wrote a Sage program implementing quantum wall crossing of $(\mathbb{P}^2, E)$.

Remark 9. *Replacing each wall in the classical scattering diagram with its quantization maintains its consistency. Recall that in the Kontsevich-Soibelman consistency lemma or Lemma 2, we compute the composition of automorphisms attached to the rays at a singular point p, and add the uniquely determined rays so that the path ordered product around p is the identity. In quantum scattering, the added rays will set-theoretically be the same, and the quantized wall functions will be the same as the classical ones, except for the appearance of powers of $\mathbf{q}$ in order to cancel out powers of $\mathbf{q}$ from the path ordered product. However, $\mathbf{q}$ is in the center of the algebra, and hence the quantum path ordered product will again be the identity.*

3.3. Example: Gross-Siebert Mirror Symmetry for $\mathbb{P}^2$

3.3.1. Context of the Gross-Siebert program

The Gross-Siebert program seeks to understand the SYZ conjecture from the viewpoint of algebraic geometry and tropical geometry. The SYZ conjecture formulates mirror symmetry as the existence of pairwise dual special Lagrangian torus fibrations. In their seminal work, Strominger, Yau, and Zaslow described this as T-duality. Let X be a

Calabi-Yau threefold that is a special Lagrangian torus fibration over an affine manifold B. Then the SYZ mirror $\check{X}$ of X is conjectured to be the dual torus fibration,

$$\begin{array}{ccc} X & & \check{X} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

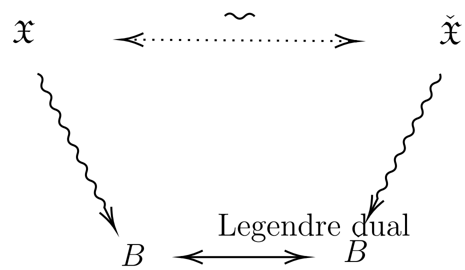
In the case of X or $\check{X}$, both the complex and symplectic structures give affine structures outside of the discriminant locus $\Delta \subset B$, and they are related to each other by a Legendre transformation. The affine structures on X and $\check{X}$ are mirror dual, in the sense that the affine structure on X from its complex structure is the affine structure on $\check{X}$ from its symplectic structure.

For Fano manifolds X with anticanonical divisor D, the SYZ conjecture predicts that the mirror $\check{X}$ is the moduli space of pairs (L, ∇), where L is a special Lagrangian torus in $X \setminus D$ and ∇ is a $U(1)$-connection on L.

$$\begin{array}{ccc} X \setminus D & & \check{X} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

In this setting, one defines a superpotential $W : \check{X} \to \mathbb{C}$ as the count of Maslov Index 2, holomorphic discs with boundary on a special Lagrangian torus fibre. As one varies the torus fibre across certain walls, the resulting count of discs is modified by additional Maslov index 0 discs, and thus the superpotential undergoes wall crossing.

From a Fano manifold X and anticanonical divisor D, Gross and Siebert consider a *toric degeneration* $\mathfrak{X} \to \operatorname{Spec} k[[t]]$, which is a degeneration of X to a singular union of toric varieties glued along toric divisors. The intersection complex B is defined as the singular fiber of $\mathfrak{X}$. It is an integral affine manifold, and one can take its *Legendre dual* to obtain the dual intersection complex $\check{B}$. From $\check{B}$, they associate a *consistent* *scattering diagram*, which determines the mirror toric degeneration. The philosophy of Gross-Siebert mirror symmetry may be summarized in the phrase *mirror dual manifolds admit degenerations with dual intersection complexes.*



Constructing the mirror toric degeneration $\hat{\mathfrak{X}}$ from the data of $\check{B}$ is the subject of [**GS13**].

3.3.2. Toric Degeneration

Let $E \in |-K_{\mathbb{P}^2}|$ be an anticanonical divisor given by a smooth elliptic curve. We consider a toric degeneration $\mathfrak{X} \to \mathbb{A}^1$ of $(\mathbb{P}^2, E)$ given by the following space,

$$\mathfrak{X} := \{XYZ - t(W + sf) = 0\} \subset \mathbb{P}(1,1,1,3) \times \mathbb{A}^2$$

where $[X : Y : Z : W]$ are coordinates of the weighted projective plane $\mathbb{P}(1,1,1,3)$, determined by the equivalence $[X : Y : Z : W] \sim [\lambda X : \lambda Y : \lambda Z : \lambda^3 W]$ for $\lambda \in \mathbb{C}^*$, and (s,t) coordinates on $\mathbb{A}^2$. Here f is a generic, degree 3, homogeneous polynomial in variables $[X : Y : Z] \in \mathbb{P}^2$. We consider the divisor $\mathfrak{D} := \{W = 0\} \subset \mathfrak{X}$.

When $s = 0, t \neq 0$, we have the equation $tW = XYZ$ and $(\mathfrak{X}, \mathfrak{D})$ is $\mathbb{P}^2$ with a triangle of lines. When $s \neq 0, t \neq 0$, then $(\mathfrak{X}, \mathfrak{D})$ is $\mathbb{P}^2$ with a smooth elliptic curve. When $t = 0$, then $(\mathfrak{X}, \mathfrak{D})$ is 3 copies of $\mathbb{P}(1,1,3)$ glued along toric divisors and a triangle of lines. The central fiber $\mathfrak{X}_{s=t=0}$ is a union of 3 $\mathbb{P}(1,1,3)$'s glued along their toric boundaries, with divisor that is a triangle of lines, and is the intersection complex associated to $\mathfrak{X}$. Taking $s = t$, we have a degeneration of $(\mathbb{P}^2, E)$ to the central fiber.

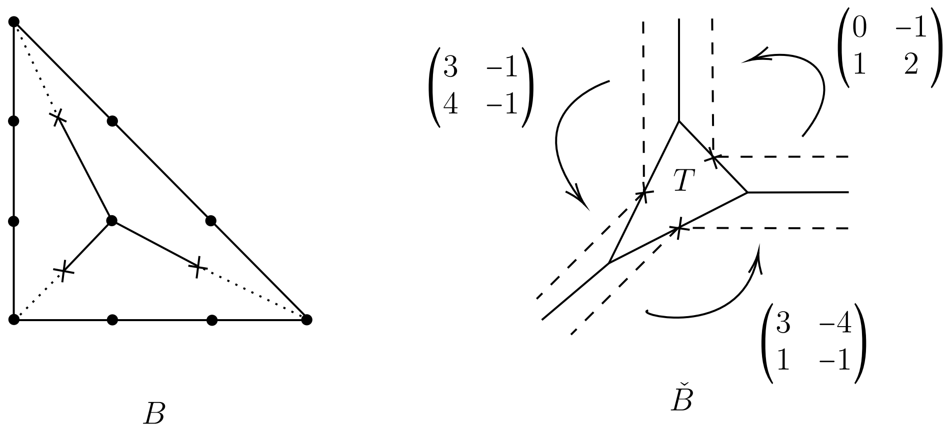


Figure 3.3. The intersection complex B of the toric degeneration $\mathfrak{X}$ of $(\mathbb{P}^2, E)$ on the left. It is a singular toric variety that is the union of 3 copies of weighted projective spaces $\mathbb{P}(1,1,3)$'s. We introduce affine singularities x's along the toric divisors of the $\mathbb{P}(1,1,3)$. Its dual $\check{B}$ is given on the right, with central chamber T and specified monodromy.

3.3.3. Intersection complexes

The central fiber of $\mathfrak{X}$ can be described as a *Fano polytope* Q, i.e. a polytope in $\mathbb{R}^2$ with vertices at integer points, and a polyhedral decomposition $\mathcal{P}$ of $\mathcal{B}$ obtained by connecting the vertices of Q to the origin. One also takes a polarization φ on Q, i.e. a strictly convex, piecewise affine function $\varphi : Q \to \mathbb{R}$ defined by $\varphi(0) = 0$ and $\varphi(v) = 1$ for vertices v of Q. We call the data $(\mathcal{B}, \mathcal{P}, \varphi)$ *the intersection complex.*

The *dual intersection complex* $(\check{B}, \check{P}, \check{\varphi})$ is obtained by taking the *Legendre dual* of $(\mathcal{B}, \mathcal{P}, \varphi)$ (see [**Gro**] for more details). It is an integral affine manifold, i.e. it has transition functions in $M \rtimes GL(M)$, from the embedding $\check{B} \subset \mathbb{R}^2$ and inheriting the ambient integral affine structure. Consider the cones $C_1 = (\frac{1}{2}, \frac{1}{2}) + (\mathbb{R}_{\geq 0}(1,0) \times \mathbb{R}_{\geq 0}(0,1))$, $C_2 = (\frac{-1}{2}, 0) + (\mathbb{R}_{\geq 0}(-1,-1) \times \mathbb{R}_{\geq 0}(0,1))$ and $C_3 = (0, \frac{-1}{2}) + (\mathbb{R}_{\geq 0}(1,0) \times \mathbb{R}_{\geq 0}(-1,-1))$. Then as a set, $\check{B} = \mathbb{R}^2 \smallsetminus \cup_i C_i$.

We introduce *affine singularities* to $\check{B}$, which we denote by x's. These are points for which we specify a non-trivial monodromy of the affine structure. By introducing monodromy, we identify,

- $(\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(1,0)$ with $(\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(0,1)$ by the $SL_2(\mathbb{Z})$-transformation $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$
- $(\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(1,0)$ with $(\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(0,1)$ by the $SL_2(\mathbb{Z})$-transformation $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$
- $(\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(1,0)$ with $(\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(0,1)$ by the $SL_2(\mathbb{Z})$-transformation $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

The gluing of the affine structure in $\check{B}$ makes the unbounded edges parallel to each other. The dual intersection complex $\check{B}$ of $\mathbb{P}^2$ is an example of an *asymptotically cylindrical* affine manifold, i.e. it has a single unbounded direction. We call the maximal two-cell that is the triangle T of $\check{B}$ the *central chamber.*

We will view the dual intersection complex $\check{B}$ in a different chart in which the unbounded edges appear parallel by "unfolding the polytope" (see [**Gra**], Definition 1 for more detail). In this chart, $\check{B}$ is equivalently the universal cover of $\tilde{U} := \check{B} - T$, where T is the maximal 2-cell that is the central triangle. The dual intersection complex in this chart appears as in Figure 3.4.

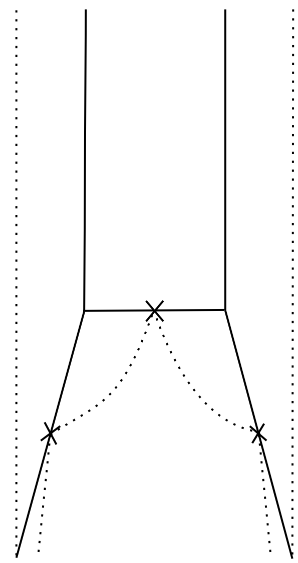


Figure 3.4. The unfolding of $\check{B}$, with an unbounded vertical direction. The region below the non-vertical dotted lines is excluded.

3.3.4. Scattering diagram of $(\mathbb{P}^2, E)$

We refer to Section 3.1 for more general details on scattering diagrams. The initial scattering diagram $\mathfrak{D}_0$ associated to the dual intersection complex $\check{B}$ has 6 rays, each emanating from an affine singularity. They are given by,

(1) $\sigma_1 = (\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(-1, 1)$ and $\sigma_2 = (\frac{1}{2}, \frac{1}{2}) + \mathbb{R}_{\geq 0}(1, -1)$

(2) $\sigma_3 = (\frac{-1}{2}, 0) + \mathbb{R}_{\geq 0}(1, 2)$ and $\sigma_4 = (\frac{-1}{2}, 0) + \mathbb{R}_{\geq 0}(-1, -2)$

(3) $\sigma_5 = (0, \frac{-1}{2}) + \mathbb{R}_{\geq 0}(2, 1)$ and $\sigma_6 = (0, \frac{-1}{2}) + \mathbb{R}_{\geq 0}(-2, -1)$

The wall functions attached to the initial rays are respectively given by,

(1) $f_1 = 1 + z^{(1,-1)}$ and $f_2 = 1 + z^{(-1,1)}$

(2) $f_3 = 1 + z^{(-1,-2)}$ and $f_4 = 1 + z^{(1,2)}$

(3) $f_5 = 1 + z^{(-2,-1)}$ and $f_6 = 1 + z^{(2,1)}$

The functions f_i are examples of *slabs*, which are the walls of the initial scattering diagram that start at affine singularities ([**Gra**], Definition 5.9).

When viewing $\mathfrak{D}_0$ in $\tilde{U}$, the initial rays can be characterized as the tangent lines of the parabola $y = \frac{-x^2}{2}$ at the integral points $(n, \frac{-n^2}{2})$. The rays of $\mathfrak{D}_0$ and of any of its consistent completions never enter into the region $\{y < \frac{-x^2}{2}\}$. From $\mathfrak{D}_0$, we inductively apply the Kontsevich-Soibelman consistency lemma (Lemma 2) to obtain a consistent scattering diagram $S(\mathfrak{D})$. If one travels sufficiently high up in Figure 3.4 (for every finite order t-cutoff), there are unbounded chambers $\mathfrak{u}$ of $S(\mathfrak{D})$. See Figure 3.5 for the scattering diagram consistent to order t^{12}.

3.3.5. Broken Lines

In the Gross-Siebert program, broken lines are piecewise linear maps used to probe the combinatorial structure of the scattering diagram. Broken lines in tropical geometry are the analogous definition of holomorphic disc counts with boundary on a moment fiber in symplectic geometry. Cluster-type wall functions of the scattering diagram capture instanton corrections to the disc count when moving across chambers.

Definition 34. *A broken line is a piecewise linear map $\beta : (-\infty, 0] \to \check{B}$ satisfying the following conditions:*

(1) There exists $t_0 = -\infty < t_1 < \ldots < t_n < 0 = t_{n+1}$ corresponding to breakpoints $\beta(t_i) \in Supp(\mathfrak{D}) \setminus Sing(\mathfrak{D})$ where β hits a wall σ_i for $1 \leq i \leq n$ of the scattering diagram. Let θ_i be the wall automorphism of σ_i. Writing $b = \bigcup_{i=0}^{n} b_i$ in terms of its piecewise linear components, each line segment b_i for $1 \leq i \leq n$ carries a term $c_i z^{m_i} \in \mathbb{C}[M] \hat{\otimes}_{\mathbb{C}} R$, where $c_i z^{m_i}$ is defined inductively as a term appearing

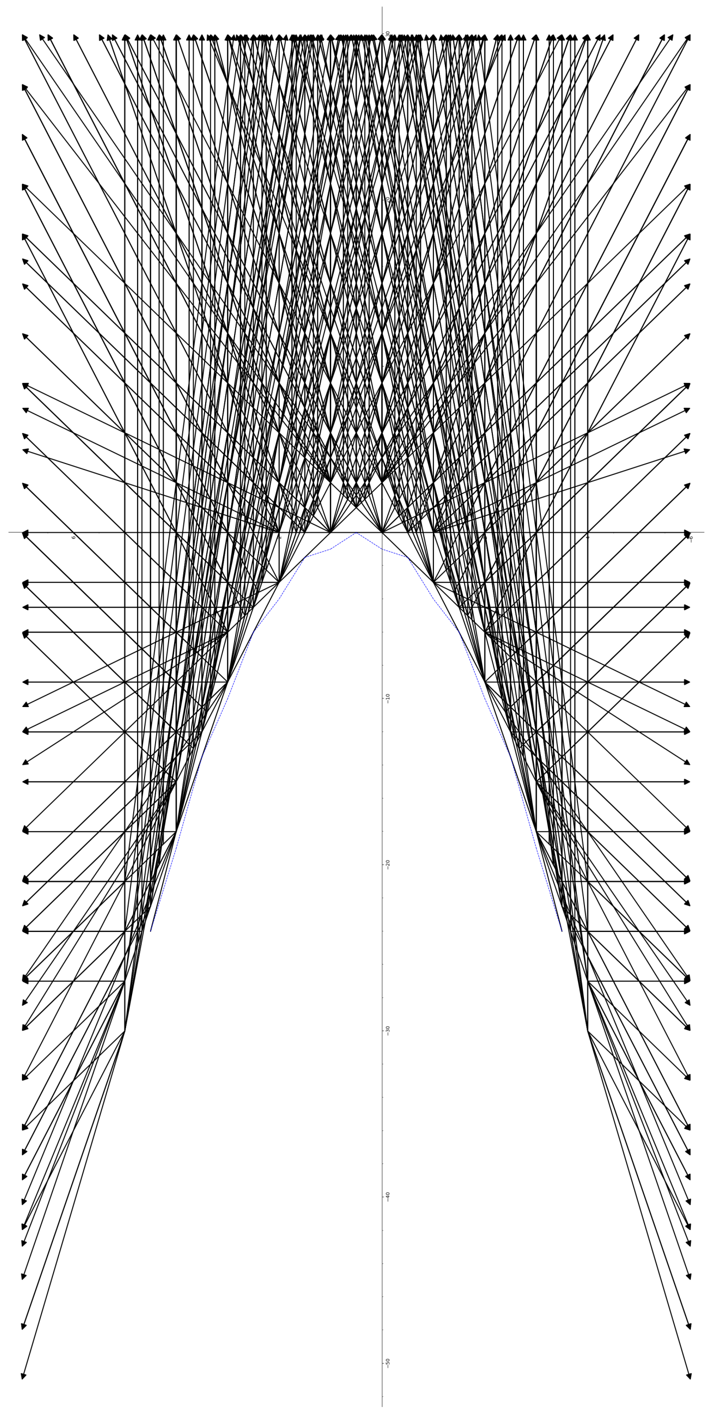


Figure 3.5. Scattering diagram of $(\mathbb{P}^2, E)$ consistent to order t^{12}, as viewed in the unfolding of $\check{B}$ (unbounded chambers not visible). This was produced by the Sage code of Tim Gräfnitz.

in the wall cross $\theta_{\sigma_{i-1}}(c_{i-1}z^{m_{i-1}})$. *For* $i = 0$, *we have* $c_0 z^{m_0}$ *with* m_0 *parallel to an unbounded direction of* $\check{B}$.

(2) *For* $0 \leq i \leq n$ *and* $t \in (t_i, t_{i+1})$, $m_i = c\beta'(t)$ *for* $c \leq 0$.

The broken line ends at a point $p \in \check{B}$ *if* $\beta(0) = p$. *For the monomial of* β, *we define* $c_\beta := c_n$ *and* $m_\beta = m_n$. *When* $R = \mathbb{C}[[t]]$, *we say that the* $t-$*order of the broken line is the power of* t *in the coefficient* c_β.

By adding hats to the z*'s and wall crossing in the quantum scattering diagram* $\hat{\mathfrak{D}}$, *we obtain the definition of a quantum broken line. We write* $c_\beta(q)$ *instead of* c_β.

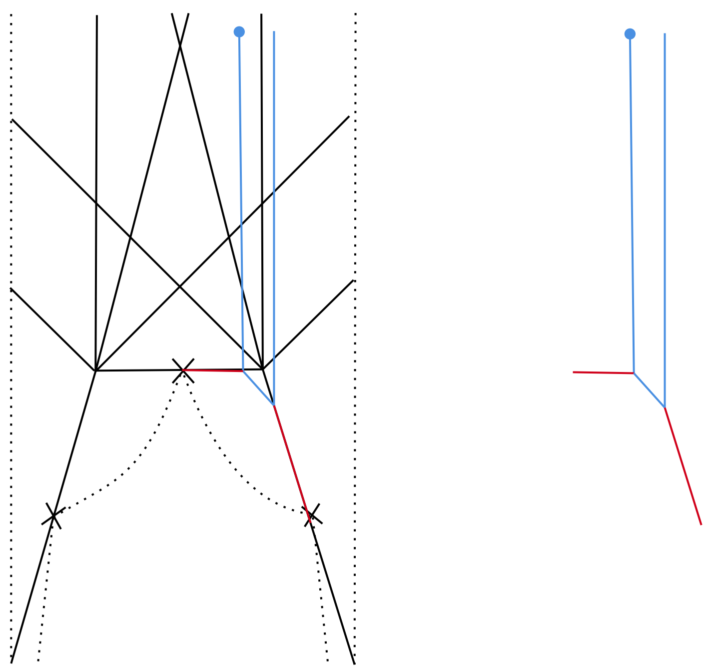


Figure 3.6. Example of a broken line (blue) and a tropical completion of it with the added walls in red.

3.3.5.1. Tropical invariants from broken lines. There is a process called *tropical completion* that produces a tropical curve from a broken line, and vice versa. Roughly, by adding the wall and its ancestors at each of the breakpoints of a broken line, one obtains a tropical curve. By the ancestors of a wall, we mean other walls involved in producing the wall from application of Lemma 2. We refer to Proposition 3.2 of [**Gra3**] for more details.

Recall that for $\mathbb{P}^2$, there is only a single unbounded direction. In $\tilde{U}$, the unbounded direction is the y-direction. For this section, let P be a general point in an unbounded chamber $\mathfrak{u}$. Broken lines with endpoint p will necessarily have an ending monomial that is parallel to the asymptotic direction ([**GRZ**], Proposition 3.5). Define $\mathfrak{B}_{p,q}(P \in \mathfrak{u})$ to be the set of broken lines ending at P with initial monomial of the form $c_0 y^q$ and ending monomial $c_n y^{-p}$, for some $c_0, c_n \in R$. Let $\mathfrak{T}_{p,q}(X, \beta, P)$ be the set of tropical curves with 2 unbounded legs of weight p and q of an effective class β, with the image of the former passing through P. We consider these sets in the scattering diagram $\mathfrak{D}_{p+q}$. Since there are finitely many walls in $\mathfrak{D}_{p+q}$, each of the sets $\mathfrak{B}_{p,q}(P \in \mathfrak{u})$ and $\mathfrak{T}_{p,q}(X, \beta, P)$ is finite.

We have the following proposition,

Proposition 2 ([**GRZ**]). *There is a finite, surjective map*

$$\mu : \coprod_{\beta} \mathfrak{T}_{p,q}(X, \beta, P) \to \mathfrak{B}_{p,q}(P \in \mathfrak{u})$$

such that for each $\beta \in \mathfrak{B}_{p,q}(P \in \mathfrak{u})$*, the preimage* $\mu^{-1}(\beta)$ *is finite and contained in a single component* $\mathfrak{T}_{p,q}(X, \beta, P)$*, and we have the equality,*

$$c_\beta(q) = \sum_{h \in \mu^{-1}(b)} m_h(q)$$

Proof. See [**GRZ**], Proposition 4.11 or a similar proof in [**Gro**], Proposition 5.32. □

Definition 35. *We define,*

$$R^{trop}_{(p,q)}(X,\beta,\mathbf{q}) := \sum_{h\in\mathfrak{T}_{p,q}(X,\beta,P)} m_h(\mathbf{q})$$

This is the count of two-legged tropical curves with 2 unbounded legs of weight p and q of an effective class β.

Let $R^{trop}_{g,(p,q)}(X,\beta)$ be the quantity,

$$R^{trop}_{(p,q)}(X,\beta,\mathbf{q}) = \sum_{g\geq 0} R^{trop}_{g,(p,q)}(X,\beta)\hbar^{2g}$$

which is determined by $R^{trop}_{(p,q)}(X,\beta,\mathbf{q})$ after the substitution $\mathbf{q} = e^{i\hbar}$ into $m_h(\mathbf{q})$.

3.3.6. Theta functions

For this section, we assume that elements cz^m live in $\mathbb{C}[M]\hat{\otimes}_{\mathbb{C}}\mathbb{C}[[t]]$. Recall that $S(\mathfrak{D}) = \mathfrak{D}_\infty$ is the direct limit of the $\mathfrak{D}_k$ with $\mathfrak{D}_k \subset \mathfrak{D}_{k+1}$. Let $\mathfrak{B}^{(k)}_q(P\in\mathfrak{u})$ be the set of broken lines with t-order k ending at P with asymptotic monomial z^q.

Definition 36. *For $k\geq 0$ and a chamber $\mathfrak{u}\in\mathfrak{D}_k$, let*

$$\theta^{(k)}_q(\mathfrak{u},\mathbf{q}) := \sum_{\beta\in\mathfrak{B}^{(k)}_q(P\in\mathfrak{u})} c_\beta(\mathbf{q})t^{d_\beta}z^{m_\beta}$$

where d_β is the t-order of the broken line β. For a nested sequence of chambers $(\mathfrak{u}_k)_{k\in\mathbb{N}}$ with $\mathfrak{u}_{k+1}\subset\mathfrak{u}_k$, define the quantum theta function $\theta_q((\mathfrak{u}_k)_{k\in\mathbb{N}},\mathbf{q})$ in the asymptotic direction q to be,

$$\theta_q((\mathfrak{u}_k)_{k\in\mathbb{N}},\mathbf{q}) := \sum_{k\geq 0}\theta^{(k)}_q(\mathfrak{u}_k,\mathbf{q}).$$

If we take $(\mathfrak{u}_k)_{k\in\mathbb{N}}$ to be a nested sequence of unbounded chambers, then by Proposition 2, we may re-express θ_q as,

$$\theta_q((\mathfrak{u}_k)_{k\in\mathbb{N}},\mathbf{q}) = y^q + \sum_{p\geq 1}\sum_{\beta|\beta\cdot D=p+q} R^{trop}_{g,(p,q)}(X,\beta,\mathbf{q})s^{\beta}t^{\deg(\beta)}y^{-p}$$

For $\mathbb{P}^2$, the asymptotic direction m is necessarily a multiple of y, since $\check{B}$ is asymptotically cylindrical. By letting $\mathbf{q}\to 1$, we recover classical theta functions.

For a given chamber $\mathfrak{u}$, the functions $\theta_q^{(k)}$ do not depend on the choice of endpoint $P\in\mathfrak{u}$. Consistency of the scattering diagram implies that the theta functions form globally well-defined functions on the dual intersection complex: we have the following theorem.

Theorem 13. *Let P,P' be two general points in $\mathfrak{D}_l$ for some $l>0$. Suppose that $(\mathfrak{u}_k)_{k\in\mathbb{N}}$ and $(\mathfrak{u}'_k)_{k\in\mathbb{N}}$ are two nested sequences of chambers of $\mathfrak{D}_l$ such that $P\in\mathfrak{u}_k$ and $P'\in\mathfrak{u}'_k$ for all k. Let $\gamma\subset M_{\mathbb{R}}\setminus Sing(S(\mathfrak{D}_l))$ be a path connecting P to P'. Then,*

$$\theta_q((\mathfrak{u}_k)'_{k\in\mathbb{N}},\mathbf{q}) = \hat{\Phi}_{\gamma}\theta_q((\mathfrak{u}_k)_{k\in\mathbb{N}},\mathbf{q})$$

Proof. See [**Gro**], Theorem 5.35 for a proof in the classical case or [**Man2**] for the $\mathbf{q}$-refined setting. □

For $q=1$, θ_1 is the *Landau-Ginzburg superpotential* W. For $\mathbb{P}^2$, there are exactly 3 broken lines with endpoint in the central chamber, with ending monomials $z^{(1,0)}$, $z^{(0,1)}$, and $z^{(-1,-1)}$. Following [**CPS**], the superpotential takes the form,

$$W = x + y + \frac{1}{xy}$$

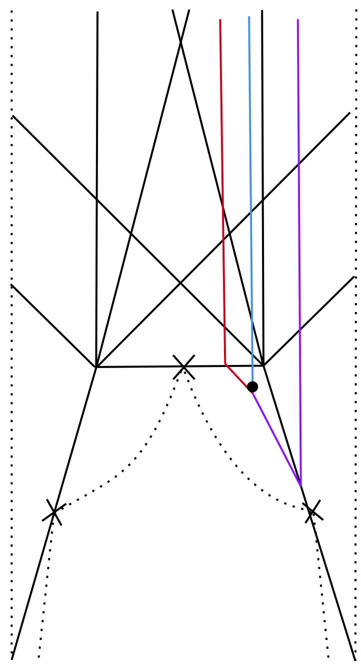


Figure 3.7. The broken lines contributing to the Landau-Ginzburg potential W in the central chamber of $(\mathbb{P}^2, E)$. The ending monomials are y, $\frac{y}{x}$ and $\frac{x}{y^2}$, hence $W = y + \frac{y}{x} + \frac{x}{y^2}$ in the central chamber. This is equivalent to the usual form of the Hori-Vafa potential after applying the $SL_2(\mathbb{Z})$-transformation $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

This definition gives the Hori-Vafa superpotential, and agrees with Cho and Oh's classification of families holomorphic discs with boundary on a moment fiber [**CO**]. The superpotential W is proper when the anti-canonical divisor is smooth [**CPS**].

Notice that W has a quantization $W(\mathbf{q})$ defined by counting mulitplicity of quantum broken lines with endpoint in a given chamber.

3.3.7. Algebra of Theta functions

In Gross, Hacking, and Keel, theta functions form a canonical basis of an algebra, whose spectrum is the mirror toric degeneration of a log Calabi-Yau pair (X, D) [**GHK**]. Here we define the algebra of theta functions.

We fix a sequence of chambers $(\mathfrak{u}_k)_{k\in\mathbb{N}}$ with $P \in \mathfrak{u}_k$. The set of theta functions with endpoint in $\mathfrak{u}$ over all asymptotic directions forms an algebra,

$$A = \bigoplus_q \mathbb{C} \cdot \theta_q((\mathfrak{u}_k)_{k\in\mathbb{N}}, \mathbf{q})$$

with multiplication given by,

$$\theta_{q_1}((\mathfrak{u}_k)_{k\in\mathbb{N}}, \mathbf{q}) \cdot \theta_{q_2}((\mathfrak{u}_k)_{k\in\mathbb{N}}, \mathbf{q}) = \sum_q \alpha^q_{q_1 q_2}(\mathbf{q}) \theta_q((\mathfrak{u}_k)_{k\in\mathbb{N}}, \mathbf{q})$$

where the structure constants $\alpha^q_{q_1 q_2}(\mathbf{q}) \coloneqq \sum_{(\beta_1,\beta_2)} c_{\beta_1}(\mathbf{q}) c_{\beta_2}(\mathbf{q})$ is a sum over all pairs of broken lines (β_1, β_2) with asymptotic directions q_1, q_2, satisfying $q_1 + q_2 = q$. It is shown in [**GRZ**], Proposition 5.2 that the following equality holds in unbounded chambers,

$$\alpha^q_{q_1 q_2}(\mathbf{q}) = \sum_{\beta | \beta\cdot D = q_1 + q_2 - q} (R^{trop}_{q_1 - q, q_2}(\beta, \mathbf{q}) + R^{trop}_{q_2 - q, q_1}(\beta, \mathbf{q}) s^\beta t^{\deg(\beta)} \tag{3.3}$$

By [**Gra**], the structure constants $\alpha^q_{q_1 q_2}(\mathbf{q})$ can then be expressed in terms of two-pointed log invariants of $X(\log D)$.

These structure constants can be also be expressed in terms of punctured Gromov-Witten invariants $N^\beta_{q_1 q_2 q}$. In [**Wan**], a formula is proven that expresses $N^\beta_{q_1 q_2 q}$ as a sum of two-pointed log invariants $R_{q_1 - q, q_2}$ and $R_{q_2 - q, q_1}$ of $X(\log D)$ similar to Equation 3.3.

Remark 10. *In* [**Gro2**]*, it is shown that mirror symmetry for* $\mathbb{P}^2$ *is equivalent to tropical formulas for descendant Gromov-Witten invariants, and certain oscillatory integrals with tropically defined Givental* J*-functions.*

Remark 11. *In a similar vein, Gross, Hacking, and Keel implemented the Gross-Siebert program for log Calabi-Yau surfaces* X *with normal crossings anti-canonical divisor* D*. The mirror family to* (X, D) *is spectrum of the algebra of theta functions with multiplication defined by genus 0 log Gromov-Witten invariants with maximal tangency to* D*.*

CHAPTER 4

Definition of Invariants

The purpose of this chapter is to establish the setting and define the invariants used in the ensuing chapters. Recall that X is a *log Calabi-Yau surface*, which is a smooth projective complex surface with an anticanonical divisor $E \in |-K_X|$. We assume that E is smooth. By the adjunction formula, E is a smooth elliptic curve. Let $H_2^+(X,\mathbb{Z})$ be the monoid of effective curve classes of X, and we take $\beta \in H_2^+(X,\mathbb{Z})$. We let $\pi : \hat{X} \to X$ be the blow up of X at a point.

For Conjectures 6, 4, 9, we will require an additional assumption that X be toric, and we will equivalently characterize X as a toric Fano surface. Then, K_X is a toric Calabi-Yau threefold. We also take $\pi : \hat{X} \to X$ to be a toric blow up. We write $X(\log E)$ to be the log scheme X with divisorial log structure given by E. Let K_X be the Calabi-Yau threefold that is the canonical bundle of X, and $Z := \mathbb{P}(K_X \oplus \mathcal{O}_X)$ be its projective compactification. We will take $\mathbf{q} = e^{i\hbar}$ as a formal variable.

4.1. Logarithmic Gromov-Witten invariants

We define two kinds of logarithmic Gromov-Witten invariants from the pair $X(\log E)$. Let $\overline{\mathcal{M}}_{g,2}(X(\log E),\beta)_{(p,q)}$ be the moduli space of genus g, basic stable log maps with 2 relative marked points to $X(\log E)$ in the curve class β, where the first point intersects a fixed point of E with contact order p, and the second point intersects a varying point of E with contact order q, such that $\beta \cdot E = p + q$. (see Section 2.6 for more details on the stable log moduli space). Let $ev = ev_1 \times ev_2 : \overline{\mathcal{M}}_{g,2}(X(\log E), \beta)_{(p,q)} \to E \times E$ be the evaluation maps of the first and second relative marked points. By the Riemann-Roch formula, the virtual dimension is given by the formula.

$$(\dim X - 3)(1-g) + \int_\beta c_1(TX) + 2 - \beta \cdot E$$

and hence is $g+1$.

We define the invariant $R_{g,(p,q)}(X(\log E), \beta)$ by inserting a λ_g-class and fixing a point $[pt] \in H^2(E)$,

$$R_{g,(p,q)}(X(\log E), \beta) := \int_{[\overline{\mathcal{M}}_{g,2}(X(\log E),\beta)_{(p,q)}]^{vir}} (-1)^g \lambda_g ev_1^*[pt] \tag{4.1}$$

Now, let $\overline{\mathcal{M}}_{g,1}(X(\log E), \beta)$ be the moduli space of genus g, basic stable log maps to $X(\log E)$ in the curve class β with 1 relative marked point of maximal tangency of order $\beta \cdot E$. Its virtual dimension is g. We define the invariant,

$$R_{g,(\beta \cdot E)}(X(\log E), \beta) := \int_{[\overline{\mathcal{M}}_{g,1}(X(\log E),\beta)]^{vir}} (-1)^g \lambda_g \tag{4.2}$$

We will sometimes suppress the log structure and curve class for these invariants and write $R_{g,(p,q)}(X(\log E), \beta)$ and $R_{g,(\beta\cdot E)}(X(\log E), \beta))$ as $R_{g,(p,q)}(X)$ and $R_{g,(\beta\cdot E)}(X)$, respectively.

4.2. Local Gromov-Witten invariants

We review the definition of local Gromov-Witten invariants. A genus g stable map $f : C \to K_X$ in curve class β is the same data as a map $f : C \to X$ in class β and a section $s : C \to f^*K_X$. Since E is ample and in particular nef, we have $E \cdot \beta > 0$ and hence $f^*K_X = f^*\mathcal{O}(-E)$ is a negative line bundle over C. Therefore, genus g stable maps $f : C \to K_X$ must factor through the zero-section X. Let $\overline{\mathcal{M}}_g(X, \beta)$ be the moduli space of genus g, unmarked stable maps to X in class β. We have the equality of moduli spaces $\overline{\mathcal{M}}_g(K_X, \beta) = \overline{\mathcal{M}}_g(X, \beta)$ by the reasoning above, however their respective obstruction theories will differ. Let $\pi : U \to \overline{\mathcal{M}}_g(X, \beta)$ be the universal curve, and $ev : U \to X$ be the universal map. The genus g local Gromov-Witten invariant of X is defined as,

$$N_g(K_X, \beta) := \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} e(R^1\pi_* ev^* K_X) \tag{4.3}$$

From comparing the obstruction theories of the surface and the threefold, this invariant is equal to,

$$\int_{[\overline{\mathcal{M}}_g(K_X,\beta)]^{vir}} 1$$

These invariants were first introduced in [**CKYZ**] and considered in higher genus [**KZ**].

4.2.1. The (closed) Gopakumar-Vafa formula

For a Calabi-Yau threefold Y, Gopakumar and Vafa conjectured that the Gromov-Witten invariants $N_g(Y, \beta)$ in curve class β can be re-expressed by integer counts of BPS states supported on holomorphic curves [**GV1**], [**GV2**]. Their re-summation formula takes into account multiple cover contributions to $N_g(Y,\beta)$, and is the following,

$$\sum_{g\geq 0} N_g(Y,\beta)\hbar^{2g-2} = \sum_{\beta=k\beta'}\sum_{g\geq 0} n_g(Y,\beta')\frac{1}{k}\left(2\sin\frac{k\hbar}{2}\right)^{2g-2} \tag{4.4}$$

We call the $n_g(Y,\beta)$ *Gopakumar-Vafa invariants* or *closed BPS invariants*, and are defined by Equation 4.4.

The $N_g(Y,\beta)$ are determined uniquely by the $n_g(Y,\beta)$ and vice versa, by Möbius inversion. The $n_g(Y,\beta')$ are conjectured to be integers and satisfy the vanishing properties $n_g(Y,\beta') = 0$ for $g >> 0$. When Y is compact, the conjecture was initially proven by Ionel and Parker [**IP18**] using symplectic methods. When Y is a toric Calabi-Yau, it was proven by Peng [**P**] and Konishi [**Kon**] using methods from the topological vertex.

4.3. Open Gromov-Witten invariants

Open Gromov-Witten invariants are virtual counts of genus g Riemann surfaces with h holes ending on specified Lagrangian submanifolds in a toric Calabi-Yau threefold. Defining a virtual class on the moduli of bordered Riemann surfaces is a difficult question, due to the presence of bubbling along the real co-dimension 1 boundary and non-compact target geometries. Despite this, there have been a litany of results related to open string invariatns. Graber and Zaslow matched physical predictions in the work of Aganagic and Vafa by assuming the moduli space of Riemann surfaces with boundary has a virtual class which satisfies a localization theorem [**GZ**]. Katz and Liu constructed well-defined open invariants from the moduli of J-holomorphic discs when the boundary Lagrangian carries a $U(1)$-action [**KL**]. Solomon and Tukachinsky defined open invariants from the Fukaya A_∞-algebra of the Lagrangian [**ST**] and showed they satisfied the Gromov-Witten axioms.

In informal terms, we describe briefly the approach of [**LLLZ**] of defining open invariants via stable relative maps, in order to state the multiple cover formula for open invariants. We refer to [**LLLZ**], or Section 6 of [**BBvG**] for more details.

Let Y be a Calabi-Yau threefold, and $L = L_1 \sqcup \ldots \sqcup L_s \subset Y$ a disjoint union of s Lagrangians L_i. We consider specific Lagrangians known as *Aganagic-Vafa (AV) branes.* These Lagrangians are isomorphic to $S^1 \times \mathbb{C}$, and were first considered in [**AV**], whereby they relate disc counting in a Calabi-Yau threefold with the classical Abel-Jacobi map on the mirror curve. AV-branes can be categorized as *inner* or *outer* (see [**FL**], Section 2.4).

The idea of [**LLLZ**] is to replace the geometry (Y, L) with a partially compactified geometry $(\hat{Y}, \hat{D} = \hat{D}_1 + \ldots + \hat{D}_s)$ with $K_{\hat{Y}} + \hat{D} = 0$. They consider the moduli space $\overline{\mathcal{M}}_{g,\overrightarrow{\mu}}(\hat{Y}/\hat{D}, \beta)$ of stable relative maps to $(\hat{Y}, \hat{D})$. For these maps, a *winding profile* $\overrightarrow{\mu} = (\mu^1, \ldots, \mu^s)$ is specified, which is a set of finite sequences $(\mu^i)_i$ of non-negative integers representing the contact orders of the stable relative maps with each $\hat{D}_i$. Let $|\mu^i|$ be the number of non-zero integers in μ^i, and $|\overrightarrow{\mu}| := \sum_i |\mu^i|$. We can write $\mu^i = (\mu^i_1, \ldots \mu^i_{|\mu^i|}) \in \mathbb{Z}^{|\mu^i|}_{>0}$. The contact orders μ^i of a stable relative map are analogous to boundary winding data $S^1 \hookrightarrow L$ of open Riemann surfaces ending on L. A choice of *framing* $\overrightarrow{f} = (f_1, \ldots, f_s) \in \mathbb{Z}^s_{\geq 0}$ is made for each $\hat{D}_i$ (see Section 3.2 [**AKMV**], for a discussion). The moduli space $\overline{\mathcal{M}}_{g,\overrightarrow{\mu}}(\hat{Y}/\hat{D}, \beta)$ is not proper, and hence does not carry a virtual fundamental class. However, it inherits a $T' \cong (\mathbb{C}^*)^2$ action from $\hat{X}$ with compact fixed loci. The virtual class $[\overline{\mathcal{M}}_{g,\overrightarrow{\mu}}(\hat{Y}/\hat{D}, \beta)]^{vir,T'}$ is defined by T'-virtual localization. Thus, the genus g, open

Gromov-Witten invariants $O^{Y,L,f}_{g,\beta,\overrightarrow{\mu}}$ of (Y,L) in class β and framing $\overrightarrow{f}$, with winding profile $\overrightarrow{\mu}$, are defined by,

$$O^{Y,L,f}_{g,\beta,\overrightarrow{\mu}} := \int_{[\overline{\mathcal{M}}_{g,\overrightarrow{\mu}}(\hat{Y}/\hat{D},\beta)]^{vir,T'}} \frac{e_T(B^{1,m})}{e_T(B^{2,m})} \tag{4.5}$$

where e_T denotes the T-equivariant Euler class, and $B^{1,m}$ are the moving parts of the obstruction theory.

In [**FL**], they prove a conjecture of Aganagic-Vafa related to disc counting and superpotentials with the above definition of open invariants. In particular, they recover the results of [**GZ**]. These open invariants also agree with those defined in [**KL**].

4.3.1. The (open) Gopakumar-Vafa formula

From the work of [**MV**], open Gromov-Witten invariants $O^{Y,L,f}_{g,\beta,\overrightarrow{\mu}}$ are conjectured to satisfy a re-summation formula in terms of *open BPS invariants* $n^{Y,L,f}_{g,\beta,\overrightarrow{\mu}}$. Given a finite sequence of non-negative integers μ^i, define the quantity,

$$z_{\mu^i} := |Aut(\mu^i)| \prod_{m=1}^{|\mu^i|} \mu^i_j$$

For a given curve class $\beta \in H_2(Y,L,\mathbb{Z})$, the multiple cover formula for open invariants ([**MV**], Equation 2.10) states that,

(4.6)

$$\sum_{g\geq 0} O^{Y,L,f}_{g,\beta,\vec{\mu}} \hbar^{2g-2+|\vec{\mu}|} = \frac{1}{\prod_{i=1}^{s} z_{\mu^i}} \sum_{k|\beta} \sum_{g\geq 0} (-1)^{|\vec{\mu}|+g} k^{|\vec{\mu}|-1} n^{Y,L,f}_{g,\frac{\beta}{k},\vec{\mu}} \left(2\sin\frac{k\hbar}{2}\right)^{2g-2} \prod_{i=1}^{s} \prod_{j=1}^{|\mu^i|} 2\sin\frac{\mu^i_j \hbar}{2}$$

In forthcoming chapters, we will specifically take $s = 1$ and consider $O^{K_X,L,0}_{g,\beta,(1)}$, or the genus g, open invariant of a single, outer AV-brane L in a toric canonical bundle K_X, in class β, with winding 1 and framing 0. Hence, we define,

$$O_g(K_X, \beta, 1) := O^{K_X,L,0}_{g,\beta,(1)} \tag{4.7}$$

We will consider the corresponding open BPS invariant $n^{K_X,L,0}_{g,\beta,(1)}$, and we define,

$$n^{open}_g(K_X, \beta, 1) := n^{K_X,L,0}_{g,\beta,(1)} \tag{4.8}$$

Remark 12. *In writing $O_g(K_X)$ in the subsequent chapters, we will write the curve class as $\beta + \beta_0$, with $\beta \in H_2(K_X, \mathbb{Z})$ and $\beta_0 \in H_2(K_X, L, \mathbb{Z})$. Hence, $O_g(K_X, \beta + \beta_0, 1) := O^{K_X,L,0}_{g,\beta+\beta_0,(1)}$ and similarly for n^{open}_g.*

CHAPTER 5

Higher genus local Gromov-Witten invariants from projective bundles

In this chapter, we present a new way to obtain certain Gromov-Witten invariants of toric Calabi-Yau threefolds using projective bundles. Our method differs from previous work by using logarithmic Gromov-Witten theory of log Calabi-Yau surfaces with λ_g-insertions. As a corollary, we prove a blow up formula for invariants of projective bundles. In addition, we conjecture open-closed correspondences in all genus, with proof in low degrees and all genus.

5.1. Introduction

Recall that X is a log Calabi-Yau surface with a smooth anti-canonical divisor E that is an elliptic curve. Define $Z := \mathbb{P}(K_X \oplus \mathcal{O}_X)$ be the 3-dimensional, projective compactification of the canonical bundle K_X. The effective curve classes $H_2^+(Z)$ decompose into a sum of effective classes in the base and fiber,

$$H_2^+(Z) = i_* H_2^+(X) \oplus H_2^+(\mathbb{P}^1)$$

where $i : X \hookrightarrow Z$ is the inclusion of X as the zero-section of Z, and $\mathbb{P}^1$ is a fiber of Z. Let $\beta + h \in H_2^+(Z, \mathbb{Z})$ be a curve class with $\beta \in H_2^+(X)$, and h is the generator of $H_2^+(\mathbb{P}^1)$.

Consider the moduli space of genus g, 1-pointed maps $\overline{\mathcal{M}}_{g,1}(Z, \beta + h)$ to the projective bundle Z in the curve class $\beta + h$. Its virtual dimension is,

$$(\dim Z - 3)(1 - g) + \int_{\beta+h} c_1(TZ) + 1$$

Since $c_1(TZ)(\beta) = 0$ and $\dim Z = 3$, the moduli space has virtual dimension 3. Hence, we define the closed Gromov-Witten invariant,

$$N_{g,1}(Z, \beta + h) := \int_{[\overline{\mathcal{M}}_{g,1}(Z,\beta+h)]^{vir}} ev^*[pt]$$

where $[pt] \in H^6(Z, \mathbb{Z})$ is the Poincaré dual of a point. This quantity is the virtual count of genus g closed curves in Z passing through a single point in a fiber. In our notation, we will often suppress the curve class and write $N_{g,1}(Z, \beta + h)$ as $N_{g,1}$ or $N_{g,1}(Z)$.

The invariant $N_{0,1}(Z, \beta + h)$ first appeared in [**Cha**], where it is equated with the genus 0, winding 1, open Gromov-Witten invariant of K_X with boundary on a torus fiber. More recently, $N_{0,1}(Z, \beta + h)$ was computed in [**Wan**] and shown to equal genus 0, two-pointed, logarithmic Gromov-Witten invariants of $X(\log E)$, up to a constant factor. The latter invariants appear in Gross-Siebert mirror symmetry where they express the structure constants of the intrinsic mirror ring of theta functions [**GS16**]

In this chapter, we use the degeneration formula for stable log maps [**KLR**] to establish a formula relating $N_{g,1}(Z, \beta + h)$ to higher genus, logarithmic Gromov-Witten invariants of $X(\log E)$ with λ_g-insertions for all $g \geq 0$. We explain how arguments used in [**vGGR**], [**Wan**] can be extended to higher genus as $\dim X = 2$. We then apply the higher genus log-local principle [**BFGW**] to obtain Gromov-Witten invariants of toric Calabi-Yau threefolds.

We refer to Chapter 4 for the definition of relevant invariants. Recall that $\pi : \hat{X} \to X$ is the blow up of X at a point with exceptional curve C, and $K_{\hat{X}}$ its canonical bundle. In Section 5.3, we prove our main theorem,

Theorem 14 (Theorem 1)**.** *There exists constants $c(g,\beta) \in \mathbb{Q}$ such that,*

$$\sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} N_{g,1}(Z, \beta + h) \hbar^{2g} Q^\beta = \sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} \left[c(g,\beta) n_g \left(K_{\hat{X}}, \pi^*\beta - C\right) \left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}}} \right)^{2g-2} Q^\beta \right] - \Delta^{pl}$$

where $\mathbf{q} = e^{i\hbar}$, and $n_g(K_{\hat{X}}, \pi^*\beta - C)$ is the genus g, Gopakumar-Vafa invariant of $K_{\hat{X}}$ in curve class $\pi^*\beta - C$. The discrepancy term Δ^{pl} is a function of the stationary Gromov-Witten theory of the elliptic curve and two-pointed log invariants of $X(\log E)$ defined in Equation 5.8.

The stationary theory of the elliptic curve was solved by [**OP**] and exhibits quasi-modularity. The two-pointed log invariants of $X(\log E)$ can be computed by broken line counting or quantum wall crossing (see Chapter 3 for more details).

In addition to Theorem 14, we will prove a blow up formula for the invariants $N_{g,1}(Z, \beta + h)$ using flops in Section 5.4,

Theorem 15 (Theorem 2)**.** *Let $c(g,\beta) \in \mathbb{Q}$ and Δ^{pl} be as in Theorem 14, and let $W = Bl_p Z$ be the blow up of Z at a point p on the infinity section of Z. Then, we have that,*

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g}Q^{\beta} = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\left[c(g,\beta)N_{g,0}(W,\beta+\tilde{L})\hbar^{2g}Q^{\beta}\right] - \Delta^{pl}$$

In Section 5.5, we will use Theorem 14 to conjecture an all genus correspondence between open Gromov-Witten invariants of K_X and the closed invariants $N_{g,1}(Z)$,

Conjecture 4 (Conjecture 2)**.** *Let $c(g,\beta)\in\mathbb{Q}$ and Δ^{pl} be as in Theorem 14. Furthermore, assume that X is toric, and $\pi:\hat{X}\to X$ is a toric blow up. Define $d(g,\beta):=(-1)^{g+1}c(g,\beta)$. We conjecture the following equality,*

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g}Q^{\beta} = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\left[d(g,\beta)n_g^{open}(K_X,\beta+\beta_0,1)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}}}\right)^{2g-2}Q^{\beta}\right]-\Delta^{pl}$$

where $n_g^{open}(K_X,\beta+\beta_0,1)$ is the genus g, 1-holed, winding 1, open BPS invariant in curve class $\beta+\beta_0$ of an outer Aganagic-Vafa brane L in framing 0 (see Chapter 4 for its definition).

Hence, open invariants of K_X can conjecturally be expressed in terms of closed invariants $N_{g,1}(Z,\beta+h)$, the stationary Gromov-Witten theory of the elliptic curve, and two-pointed log invariants of $X(\log E)$.

From computations in the topological vertex in Chapter 7, we prove Conjecture 4 in low degrees and all genus for $X=\mathbb{P}^2$,

Theorem 16 (Theorem 5)**.** *Let $X=\mathbb{P}^2$ and $H\in H_2(\mathbb{P}^2,\mathbb{Z})$ the hyperplane class. Then Conjecture 4 holds in curve classes $\beta=dH$ for $d=1,2,3,4$ and all genus.*

We also give explicit genus 1 formulas for Theorems 14, 15, and 16 in Corollaries 1, 2, Corollary 3, and Conjecture 5, respectively.

5.1.1. Related work in GW-theory of projective bundles

Projective bundles are of particular interest in Gromov-Witten theory, since they appear as "bubble components" in Jun Li's theory of expanded degenerations. The main issue in relative Gromov-Witten theory is that a limit of relative stable maps intersecting properly a divisor D may not intersect properly in the limit. Bubble components or projectivized normal bundles are introduced so that the limit stable map has proper intersection with D.

The Gromov-Witten theory of projective bundles has been studied in [**Fan**], where he showed that if two vector spaces V_1 and V_2 have the same Chern classes, then the Gromov-Witten theory of their projectivizations $\mathbb{P}(V_i)$ are equal. Projective bundles are toric, as they admit a natural $\mathbb{C}^*$-action that scales the fiber. Coates showed that the Virasoro constraints are satisfied for toric bundles if and only if they are satisfied for the base.

Maulik and Pandharipande study the absolute Gromov-Witten theory of hypersurfaces by deforming to the normal cone [**MP**]. The absolute and relative Gromov-Witten theory of projective bundles naturally appears, where invariants relative to the 0- or ∞-section are considered. They use localization to compute the absolute theory of projective bundles. There is a natural $\mathbb{C}^*$-action that scales the fiber. Its fixed locus are maps going into the 0- or ∞-section, both of which are isomorphic to X. Therefore, the virtual localization formula reduces the Gromov-Witten theory of Z to Hodge integrals on X.

In our work, we instead use the degeneration formula for stable log maps [**KLR**] to compute the absolute invariants $N_{g,1}(Z,\beta+h)$ by relating them to two-pointed logarithmic Gromov-Witten invariants of $X(\log E)$ with λ_g-insertions.

5.2. Degeneration of projective bundles

5.2.1. The degeneration

We take a degeneration of Z to a normal crossings singular fiber to compute the invariants $N_{g,1}$. Let $\mathscr{X} = Bl_{E\times 0}(X\times\mathbb{A}^1)$ be the deformation to the normal cone $\pi:\mathscr{X}\to\mathbb{A}^1$. The fiber $\mathscr{X}^{-1}(t)$ when $t\neq 0$ is isomorphic to X. The special or singular fiber $\mathscr{X}^{-1}(0)$ is isomorphic to $Bl_E X\sqcup_{\mathbb{P}(N_{E/X})}\mathbb{P}(N_{E/X}\oplus\mathcal{O}_E)$, as the exceptional hypersurface is the projectivization of $N_{E\times 0}(X\times\mathbb{A}^1)\cong N_{E/X}\oplus\mathcal{O}_E$. As $\mathbb{P}(N_{E/X})\cong E$, the blow up along the divisor E does not change X. Hence, the special fiber is $X\sqcup_E\mathbb{P}(N_{E/X}\oplus\mathcal{O}_E)$. Denote Y to be the exceptional hypersurface $\mathbb{P}(N_{E/X}\oplus\mathcal{O}_E)$. The two pieces are glued along the 0-section of Y, which corresponds to the summand $N_{E/X}$. Let $\pi_Y:Y\to E$ be the projection map.

Let E_0 and E_∞ be the sections of Y corresponding to the summands $N_{E/X}$ and $\mathcal{O}_E$ respectively. Let $\mathscr{E} := \overline{\pi^{-1}(E\times\mathbb{A}^1\smallsetminus E\times 0)}$ be the strict transform of $E\times\mathbb{A}^1$. Define the space $\mathcal{L}=\mathbb{P}(\mathcal{O}_{\mathscr{X}}(-\mathscr{E})\oplus\mathcal{O}_{\mathscr{X}})$ on $\mathscr{X}$, which will serve as a degeneration $\mathcal{L}\to\mathbb{A}^1$ of Z. The generic fiber $\mathcal{L}_t$ when $t\neq 0$ is isomorphic to Z, and the special fiber $\mathcal{L}_0$ is isomorphic to $X\times\mathbb{P}^1\sqcup_{E\times\mathbb{P}^1}\mathbb{P}(\mathcal{O}_Y(-E_\infty)\oplus\mathcal{O}_Y)$. We denote $\mathcal{L}_X = X\times\mathbb{P}^1, \mathcal{L}_E = E\times\mathbb{P}^1$ and $\mathcal{L}_Y=\mathbb{P}(\mathcal{O}_Y(-E_\infty)\oplus\mathcal{O}_Y)$, hence $\mathcal{L}_0=\mathcal{L}_X\sqcup_{\mathcal{L}_E}\mathcal{L}_Y$. There are natural projection maps $\pi_{\mathcal{L}}:\mathcal{L}\to\mathbb{A}^1$ and $\pi_{\mathcal{L}_Y}:\mathcal{L}_Y\to Y$. The restriction of $\mathcal{L}_Y$ onto a fiber of $Y\to E$ is the first Hirzebruch surface $\mathbb{F}_1=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1})$. See Figure 5.1.

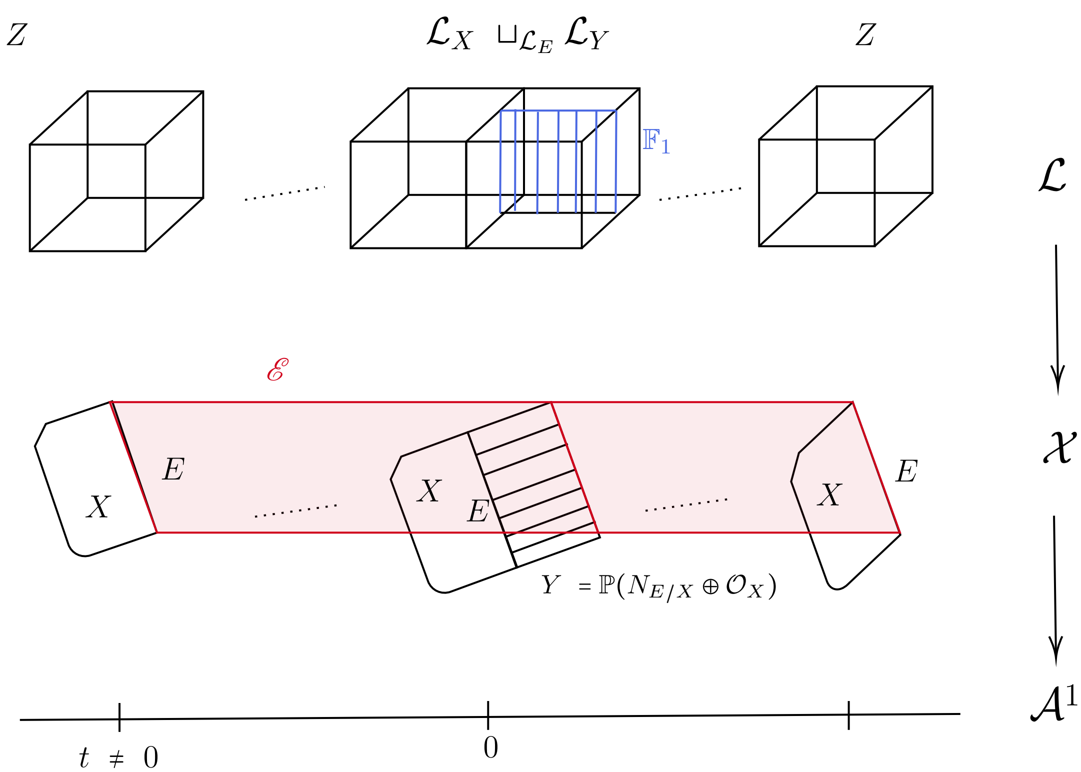


Figure 5.1. The degeneration $\mathcal{L} \to \mathscr{X} \to \mathbb{A}^1$ of the projective bundle $Z = \mathbb{P}(K_X \oplus \mathcal{O}_X)$ to the central fiber $\mathcal{L}_0 = \mathcal{L}_X \sqcup_{\mathcal{L}_E} \mathcal{L}_Y$. The space $\mathcal{L}$ is the projective bundle corresponding to the divisor $\mathscr{E}$ (shaded in red) of the deformation the normal cone $\mathscr{X} \to \mathbb{A}^1$. When restricting $\mathcal{L}_Y$ over a fiber of $Y \to E$, one obtains the first Hirzebruch surface $\mathbb{F}_1$(shaded in blue).

5.2.2. Stable log maps to $\mathcal{L}$

We consider stable log maps to the degeneration $\mathcal{L}$ of Z. We take the divisorial log structure on $\mathcal{L}$ given by the central fiber $\mathcal{L}_0$. Let $\overline{\mathcal{M}}_{g,n+r}(\mathcal{L}(\log \mathcal{L}_0), \beta + h)$ be the moduli space of genus g, basic stable log maps to $\mathcal{L}$ in the curve class $\beta+h$, with n interior marked points and r relative marked points.

We explain the notation of the curve class $\beta+h$ in $\overline{\mathcal{M}}(\mathcal{L}, \beta+h)$. Recall that $\beta \in H_2^+(X)$ and $h \in H_2^+(\mathbb{P}^1)$ is the class of a $\mathbb{P}^1$-fiber. The class $\beta + h$ strictly lives in $H_*(\mathcal{L}_t) \cong H_*(Z)$ for $t \neq 0$. When writing $\beta + h$ as a curve class in $\mathcal{L}$, we refer to a global lifting of $\beta + h$ to a class $\alpha \in H_*(\mathcal{L})$ such that $\alpha|_{\mathcal{L}_t} = \beta + h$ for all $t \neq 0$. On the central fiber $\mathcal{L}_0 = \mathcal{L}_X \sqcup_{\mathcal{L}_E} \mathcal{L}_Y$, if we decompose $\alpha|_{\mathcal{L}_0} = \beta_X + \beta_Y$ with $\beta_X \in H_*(\mathcal{L}_X), \beta_Y \in H_*(\mathcal{L}_Y)$, then $\alpha|_{\mathcal{L}_0}$ must satisfy $(\pi_{\mathcal{L}_X})_*\beta_X + (\pi_Y)_*\beta_Y = \beta + h$, where $\pi_{\mathcal{L}_X} : \mathcal{L}_X \to X$ is the projection map and $\pi_Y : Y \to E$. On curve classes β_Y in $\mathcal{L}_Y$, the map π_Y contracts fibers of $Y \to E$. Hence, for simplicity we write maps to $\mathcal{L}$ in class α as maps in class $\beta + h$.

Stable log maps to the generic fiber $\mathcal{L}_t$ will not intersect the central fiber, and hence the log structure of those stable maps is trivial. After forgetting the log structure, the stable log moduli space to $\mathcal{L}_t$ is isomorphic to the ordinary moduli space of stable maps, $\overline{\mathcal{M}}(\mathcal{L}_t(\log \mathcal{L}_E), \beta + h) \cong \overline{\mathcal{M}}(Z, \beta + h)$. We also take the divisorial log structure on $\mathbb{A}^1$ with respect to $\{0\}$. As $\mathcal{L} \to \mathbb{A}^1$ is a normal crossings degeneration, it is log smooth. By [**GS13**], the moduli space $\overline{\mathcal{M}}(\mathcal{L}/\mathbb{A}^1, \beta + h)$ is proper.

We have the following lemma (adapted from [**vGGR**], Lemma 2.2) which relates the virtual class of $\overline{\mathcal{M}}(\mathcal{L}_t)$ to the virtual class of $\overline{\mathcal{M}}(\mathcal{L}_0)$.

Lemma 4. *Let $P_0 : \overline{\mathcal{M}}(\mathcal{L}_0(\log \mathcal{L}_E), \beta + h) \to \overline{\mathcal{M}}(X, \beta)$ be the map that forgets the log structure, composes with the natural maps $\mathcal{L}_0 \to \mathscr{X}_0 \to X$, and stabilizes, and $P_t : \overline{\mathcal{M}}(Z, \beta + h) \to \overline{\mathcal{M}}(X, \beta)$ be the map that composes with the projection $Z \to X$, and stabilizes. Let $P : \overline{\mathcal{M}}(\mathcal{L}(\log \mathcal{L}_0), \beta + h) \to \overline{\mathcal{M}}(X \times \mathbb{A}^1/\mathbb{A}^1, \beta)$ be the map of moduli spaces that restricts to P_t or P_0 on each fiber. Let $\overline{\mathcal{M}}(X \times \mathbb{A}^1/\mathbb{A}^1, \beta)$ be the space of ordinary stable maps to $X \times \mathbb{A}^1$ in curve class β.*

When $t \neq 0$, we have the following equality of virtual cycles,

$$(P_0)_*[\overline{\mathcal{M}}(\mathcal{L}_0(\log \mathcal{L}_E), \beta + h)]^{vir} = (P_t)_*[\overline{\mathcal{M}}(Z, \beta + h)]^{vir}$$

Proof. We have the following commutative diagram,

$$
\begin{array}{ccccc}
\overline{\mathcal{M}}(\mathcal{L}_0(\log \mathcal{L}_E), \beta + h) & \hookrightarrow & \overline{\mathcal{M}}(\mathcal{L}(\log \mathcal{L}_0), \beta + h) & \hookleftarrow & \overline{\mathcal{M}}(Z, \beta + h) \\
\downarrow{\scriptstyle P_0} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P_t} \\
\overline{\mathcal{M}}(X, \beta) & \longrightarrow & \overline{\mathcal{M}}(X \times \mathbb{A}^1, \beta) & \longleftarrow & \overline{\mathcal{M}}(X, \beta) \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\
\{0\} & \overset{i_0}{\hookrightarrow} & \mathbb{A}^1 & \overset{i_t}{\hookleftarrow} & \{t\}
\end{array}
$$

We have the following equalities,

$$
\begin{aligned}
(P_0)_*[\overline{\mathcal{M}}(\mathcal{L}_0)]^{vir} &= (P_0)_* i_0^! [\overline{\mathcal{M}}(\mathcal{L}/\mathbb{A}^1]^{vir} \\
&= i_0^! P_* [\overline{\mathcal{M}}(\mathcal{L}/\mathbb{A}^1]^{vir} \\
&= i_t^! P_* [\overline{\mathcal{M}}(\mathcal{L}/\mathbb{A}^1]^{vir} \\
&= (P_t)_* i_t^! [\overline{\mathcal{M}}(\mathcal{L}/\mathbb{A}^1]^{vir} \\
&= (P_t)_* [\overline{\mathcal{M}}(Z, \beta + h)]^{vir}
\end{aligned}
$$

The 1st and 5th equalities follow from $[\overline{\mathcal{M}}(\mathcal{L}_0)]^{vir} = i_0^! [\overline{\mathcal{M}}(\mathcal{L}/\mathbb{A}^1)]^{vir}$ and $[\overline{\mathcal{M}}(Z)]^{vir} = i_t^! [\overline{\mathcal{M}}(\mathcal{L}/\mathbb{A}^1)]^{vir}$, which follow from the compatibility of virtual classes with base change. The 2nd and 4th equalities follow from commutativity of Gysin pullback with proper pushforward applied to the top left and right squares. The 3rd equality follows because f is the trivial family. □

5.2.3. The degeneration formula

The degeneration formula for relative Gromov-Witten invariants first appeared in work of [**Li**],[**IP**]. For stable log maps, [**KLR**] proved a degeneration formula when the singular fiber consists of two irreducible components. In [**ACGS**], a degeneration formula is proven for more general singular fibers using the technology of rigid tropical maps.

It is known that logarithmic Gromov-Witten invariants are constant in log smooth families (see Appendix A of [**MR**]): suppose that $G \to B$ is a log smooth, degenerating family, and let $i_b : G_b \hookrightarrow G$ be the inclusion of the fiber over $b \in B$ into G. Then, the invariance of logarithmic Gromov-Witten invariants is the statement that,

$$[\overline{\mathcal{M}}(G_b)]^{vir} = i_b^![\overline{\mathcal{M}}(G)]^{vir}$$

As a consequence, logarithmic Gromov-Witten invariants of the generic fiber are equal to logarithmic Gromov-Witten invariants of the singular fiber.

The degeneration formula of [**KLR**] splits the virtual class of maps to the singular fiber into virtual classes of maps to each of its irreducible components. Let $\overline{\mathcal{M}}(\mathcal{L}_0) := \overline{\mathcal{M}}_{g,n}(\mathcal{L}_0(\log \mathcal{L}_E), \beta + h)$ be the moduli space of genus g, n-marked, stable log maps to $\mathcal{L}_0(\log \mathcal{L}_E)$ of curve class $\beta + h$. Stable maps to the singular fiber are encoded by bipartite graphs Γ. Denote the vertices as $V(\Gamma)$ and edges as $E(\Gamma)$. We assign to each vertex $V \in V(\Gamma)$ a non-negative integer $g_V \geq 0$, a curve class $\beta_V \in H_2(\mathcal{L}_V)$, and a subset of markings $n_V \subset \{1, 2, \ldots, n\}$. We assign to each edge $e \in E(\Gamma)$ a non-negative integer weight $w_e \geq 0$. Edges connecting any two vertices represent relative contact orders, and no two vertices on the same side are connected by an edge. There are the following conditions on Γ can be found in Section 2 of [**KLR**],

$$i_*\beta_X + p_*\beta_y = \beta + h$$

$$1 - \chi_{top}(\Gamma) + \sum_V g_V = g$$

$$\bigcup_V n_V = \{1, 2, \dots, n\}$$

$$\sum_e w_e = \beta \cdot D$$

We denote $\Gamma(g, n, \beta)$ to be the set of all such bipartite graphs Γ satisfying the above conditions.

Vertices on one side of Γ encode moduli of stable log maps to $\mathcal{L}_X$ and vertices on the other side encode moduli of stable log maps to $\mathcal{L}_Y$. For each vertex V, let r_V be the number of edges it has. Define an index $i(V)$ to be X or Y depending on which side V lives in. Define $\overline{\mathcal{M}}_V := \overline{\mathcal{M}}_{g_V, n_V + r_V}(\mathcal{L}_{i(V)}(\log \mathcal{L}_E), \beta_V)$ to be the moduli space of genus g_V basic stable log maps to $\mathcal{L}_{i(V)}(\log \mathcal{L}_E)$ with n_V interior marked points and r_V relative marked points in class β_V. It has two natural evaluation maps of the interior or relative marked points. Let $ev_1 : \overline{\mathcal{M}}_V \to \mathcal{L}_{i(V)}^{n_V}$ and $ev_2 : \overline{\mathcal{M}}_V \to \mathcal{L}_D^{r_V}$ be the evaluation map of interior and relative marked points, respectively. We say that a vertex V is an X-vertex or Y-vertex if maps in $\overline{\mathcal{M}}_V$ map to $\mathcal{L}_X$ or $\mathcal{L}_Y$, respectively.

We have the following commutative diagram,

$$\begin{array}{ccc} \odot_V \overline{\mathcal{M}}_V & \longrightarrow & \prod_V \overline{\mathcal{M}}_V \\ \downarrow & & \downarrow{\scriptstyle ev} \\ \prod_e E & \xrightarrow{\Delta} & \prod_V \prod_{V \in e} E \times E \end{array}$$

which defines the space $\odot_V \overline{\mathcal{M}}_V$ as the fiber product. A stable map in $\odot_V \overline{\mathcal{M}}_V$ satisfies the predeformability condition (see Section 2.2 of [**GV**]); if two vertices V_1 and V_2 of Γ are joined by edge e, then maps in $\overline{\mathcal{M}}_{V_1}$ and $\overline{\mathcal{M}}_{V_2}$ will intersect at the same point in the divisor $\mathcal{L}_E$ with contact orders w_e.

Let $\overline{\mathcal{M}}_\Gamma$ be the space of stable maps whose dual intersection graph collapses to Γ with a subset of its nodes corresponding to edges $e_1, \dots, e_r$. We have an étale map that partially forgets the log structure $\Phi : \overline{\mathcal{M}}_\Gamma \to \odot_V \overline{\mathcal{M}}_V$ with $\deg \Phi = \frac{\prod_e w_e}{lcm\{w_e\}}$ (Equation 1.4 of [**KLR**]). We also have a finite map $F : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}(\mathcal{L}_0)$ that forgets the graph marking of the stable map.

The degeneration formula sums over all possible bipartite graphs $\Gamma \in \Gamma(g,n,\beta)$ to obtain the virtual class of the central fiber $\mathcal{L}_0$.

Theorem 17 ([**KLR**]). *We have the equality of virtual classes,*

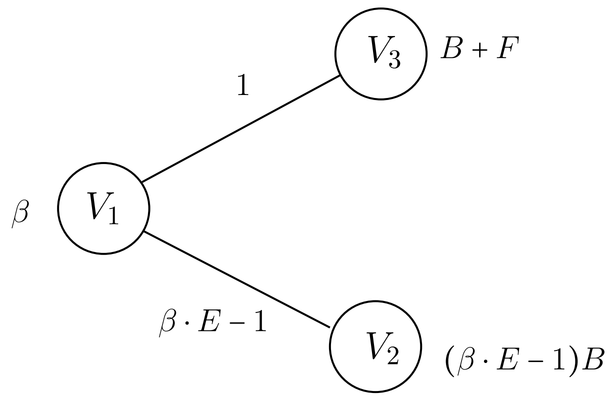
$$[\overline{\mathcal{M}}(\mathcal{L}_0)]^{vir} = \sum_{\Gamma \in \Gamma(g,n,\beta)} \frac{lcm\{w_e\}}{|Aut(\Gamma)|} F_* \Phi^* \Delta^! \prod_V [\overline{\mathcal{M}}_V]^{vir}$$

We apply the above theorem to compute the invariants $N_{g,1}(Z, \beta + h)$ of the projective bundle.

5.2.4. Bipartite graphs in the degeneration

In genus 0, the analysis for the bipartite graphs Γ that can appear is done in [**Wan**]. In order to extend to higher genus, we describe how to generalize the necessary lemmas from [**vGGR**]. We remark that they can be extended when $\dim X = 2$.

Theorem 18. *Let $g \geq 0$. The bipartite graphs Γ that have nonzero contribution in the degeneration are,*



Let B denote the fiber class of $\pi_Y : Y \to E$, and F denote the fiber class of $\pi_{\mathcal{L}_Y} : \mathcal{L}_Y \to Y$. The curve class $\beta \in H_2(X, \mathbb{Z})$ is attached to vertex V_1, the class $(\beta \cdot E - 1)B$ is attached to vertex V_2, and the class $w_{e_1}B + F$ is attached to vertex V_3. The edge connecting vertices V_1 and V_3 has weight $w_{e_1} = 1$, and the edge connecting vertices V_1 and V_2 has weight $w_{e_2} = e - 1$. We have $g = g_{V_1} + g_{V_2} + g_{V_3}$.

5.2.4.1. Condition on the X-vertices. In this section, we show that any X-vertex of Γ has at most two edges (see Section 5.2.3 for definition of X-vertex).

Lemma 5. *Let Γ be a graph with an X-vertex V with $r > 2$ adjacent edges, then* $[\overline{\mathcal{M}}_\Gamma]^{vir} = 0$.

Proof. Since non-surjective maps from a proper, genus g (nodal) curve to $\mathbb{P}^1$ are constant, the evaluation map $\overline{\mathcal{M}}_V \to (E \times \mathbb{P}^1)^{r_V}$ factors through $E^{r_V} \times \mathbb{P}^1$, where $\mathbb{P}^1$ is embedded diagonally. In addition, we separate out $\overline{\mathcal{M}}_V$ from $\prod_{V' \neq V} \overline{\mathcal{M}}_{V'}$. We have the following commutative diagram.

$$\begin{array}{ccc}
\overline{\mathcal{M}}_V \times_{\mathcal{L}_E^r} \odot_{V' \neq V} \overline{\mathcal{M}}_{V'} & \longrightarrow & \overline{\mathcal{M}}_V \times \prod_{V' \neq V} \overline{\mathcal{M}}_{V'} \\
\downarrow{\scriptstyle ev} & & \downarrow{\scriptstyle ev} \\
(E^r \times \mathbb{P}^1) \times (E \times \mathbb{P}^1)^s & \xrightarrow{\Delta'} & (E^r \times \mathbb{P}^1)^2 \times (E \times \mathbb{P}^1)^{2s} \\
\downarrow{\scriptstyle \delta := (id \times diag) \times id} & & \downarrow{\scriptstyle \delta' := (id \times diag) \times id} \\
(E \times \mathbb{P}^1)^r \times (E \times \mathbb{P}^1)^s & \xrightarrow{\Delta} & (E \times \mathbb{P}^1)^{2r} \times (E \times \mathbb{P}^1)^{2s}
\end{array}$$

Let N, N' be the normal bundles of Δ, Δ', respectively. Define $A := \delta^* N / N'$, with rank $r-1$. The excess intersection formula ([**Ful**], Theorem 6.3) tells us that,

$$\Delta^! \alpha = c_{r-1}(ev^* A) \cap (\Delta')^! \alpha$$

for $\alpha \in A_*(\overline{\mathcal{M}}_V \times \prod_{V' \neq V} \overline{\mathcal{M}}_{V'})$.

The normal bundle M of δ is $T(\mathbb{P}^1)^{r-1} = \mathcal{O}_{\mathbb{P}^1}(2)^{r-1}$, and the normal bundle M' of δ' is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2)^{2r-2}$. By the Cartesian property of the bottom square, we have that $A \cong (\Delta')^* M'/M \cong \mathcal{O}_{\mathbb{P}^1}(2)^{r-1}$. We see that $c_{r-1}(A) = 0$ if $r > 2$. Applying this to the class $[\overline{\mathcal{M}}_V]^{vir} \times \prod_{V' \neq V} [\overline{\mathcal{M}}_{V'}]^{vir}$, we have the desired result. □

5.2.4.2. Conditions on the Y-vertices. Let V be a Y-vertex. Recall that we have the projections $\pi_{\mathcal{L}_Y} : \mathcal{L}_Y \to Y$ and $\pi_Y : Y \to E$. Note that the evaluation map $ev_2 : \overline{\mathcal{M}}(\mathcal{L}_Y(\log \mathcal{L}_E), \beta_V) \to (E \times \mathbb{P}^1)^{r_V}$ can be decomposed as,

$$\overline{\mathcal{M}}(\mathcal{L}_Y(\log \mathcal{L}_E), \beta_V) \xrightarrow{\pi_{\mathcal{L}_Y}} \overline{\mathcal{M}}(Y(\log E_0), (\pi_{\mathcal{L}_Y})_* \beta_V) \xrightarrow{p} \overline{\mathcal{M}}(E, (\pi_Y \circ \pi_{\mathcal{L}_Y})_* \beta_V) \xrightarrow{ev} E \times 0 \hookrightarrow (E \times \mathbb{P}^1)^{r_V}$$

The map p forgets the log structure, composes with π_Y, and stabilizes. Since E_∞ is nef, we have $-E_\infty \cdot (\pi_{\mathcal{L}_Y})_* \beta_V < 0$, and therefore have the equality of moduli spaces

$\overline{\mathcal{M}}(Y(\log E_0), (\pi_{\mathcal{L}_Y})_*\beta_V) = \overline{\mathcal{M}}(\mathcal{L}_Y(\log \mathcal{L}_E), (\pi_{\mathcal{L}_Y})_*\beta_V)$. We first show that $(\pi_{\mathcal{L}_Y})_*\beta_V$ must be a multiple of a fiber class of $Y \to E$.

Lemma 6. *If the curve class $(\pi_{\mathcal{L}_Y})_*\beta_V$ is not a multiple of a fiber class of $Y \to E$, then $(ev_2)_*[\overline{\mathcal{M}}_V]^{vir} = 0$.*

Proof. It suffices to generalize Proposition 5.3 of [**vGGR**] to higher genus. We have the following commutative diagram,

$$\begin{array}{ccccc} \overline{\mathcal{M}}(Y(\log E_0), (\pi_{\mathcal{L}_Y})_*\beta_V) & \xrightarrow{u} & \mathscr{M} & \xrightarrow{v} & \overline{\mathcal{M}}(E, (\pi_Y \circ \pi_{\mathcal{L}_Y})_*\beta_V) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{M}^{log}_{g,n,H_2(Y)^+} & \xrightarrow{id} & \mathfrak{M}^{log}_{g,n,H_2(Y)^+} & \xrightarrow{\nu} & \mathfrak{M}_{g,n,H_2(E)^+} \end{array}$$

The space $\mathfrak{M}^{log}_{g,n,H_2(Y)^+}$ was introduced in [**Cos03**], and is the stack of genus g, n-marked, pre-stable log curves that additionally remembers the curve class of each irreducible component. Its usefulness lies in the fact that we have an isomorphism of obstruction theories $E^\bullet_{\overline{\mathcal{M}}(Y,\beta)/\mathfrak{M}^{log}_{g,n,H_2(Y)^+}} \cong E^\bullet_{\overline{\mathcal{M}}(Y,\beta)/\mathfrak{M}_{g,n}}$, since the forgetful map $\mathfrak{M}^{log}_{g,n,H_2(Y)^+} \to \mathfrak{M}_{g,n}$ is étale. The space $\mathscr{M}$ is defined to make the right hand square Cartesian, and its obstruction theory is defined as the pullback obstruction theory by ν. By the results in [**Man08**], we have,

$$\nu^![\overline{\mathcal{M}}_{g,n}(E, p_*\beta)]^{vir} = [\mathscr{M}]^{vir}$$

The above diagram is used to prove the following theorem,

Theorem 19 ([**vGGR**]). *Let $\pi_Y : Y \to E$ be a log smooth morphism where E has trivial log structure. Suppose that for every log smooth morphism $f : C \to Y$ of genus g and class β we have $H^1(C, f^*TY^{log}) = 0$, then*

$$\overline{\mathcal{M}}_{g,n}(Y(\log E_0),(\pi_{\mathcal{L}_Y})_*\beta_V) = u^*\nu^![\overline{\mathcal{M}}_{g,n}(E,(\pi_Y\circ\pi_{\mathcal{L}_Y})_*\beta)]^{vir}$$

provided that $[\overline{\mathcal{M}}_{g,n}(E,(\pi_Y\circ\pi_{\mathcal{L}_Y})_*\beta_V)]^{vir}\neq 0$. *In particular, if* $[\overline{\mathcal{M}}_{g,n}(E,(\pi_Y\circ\pi_{\mathcal{L}_Y})_*\beta_V)]^{vir}$ *can be represented by a cycle supported on some locus* $W\subset\overline{\mathcal{M}}_{g,n}(E,(\pi_Y\circ\pi_{\mathcal{L}_Y})_*\beta_V)$, *then* $[\overline{\mathcal{M}}_{g,n}(Y(\log E_0),\beta_V)]^{vir}$ *can be represented by a cycle supported on* $w^{-1}(W)$ *where* $w=v\circ u$.

The convexity assumption in Theorem 19 on $\overline{\mathcal{M}}_{g,n}(Y(\log E_0))$ guarantees that u is smooth. This implies that the virtual pullback $u^!$ defined by [**Man08**] agrees with smooth pullback u^*. For our purposes, we relax the convexity assumption in Theorem 19, which only holds in $g=0,1$. This has the effect of making u potentially non-smooth, however the virtual pullback $u^!$ is still defined. We have the short exact sequence,

$$0\to T(Y(\log E_0)/E_0)^{log}\to T(Y(\log E_0))^{log}\to TE_0\to 0$$

where $T(Y(\log E_0)/E_0)^{log}$ is the logarithmic tangent bundle of $Y(\log E_0)$ relative to E_0. It induces a compatible triple for the left hand square of the above commutative diagram. Thus, applying [**Man08**], Corollary 4.9, the virtual classes are now related by,

$$[\overline{\mathcal{M}}_{g,n}(Y(\log E_0),(\pi_{\mathcal{L}_Y})_*\beta_V)]^{vir} = u^!\nu^![\overline{\mathcal{M}}_{g,n}(E,(\pi_Y\circ\pi_{\mathcal{L}_Y})_*\beta)]^{vir}$$

In particular, if $[\overline{\mathcal{M}}(E,\pi_{Y*}\beta)]^{vir}$ is a cycle supported on some locus W, then $[\overline{\mathcal{M}}(Z,\beta)]^{vir}$ is a cycle supported on $v^{-1}u^{-1}(W)$. This latter detail allows Lemma 6 and consequently Proposition 5.3 of [**vGGR**] to be generalized to higher genus, which implies

$p_*[\overline{\mathcal{M}}(Y(\log E_0),(\pi_{\mathcal{L}_Y})_*\beta_V)]^{vir} = 0$. This gives the desired vanishing, and we have the proof of Lemma 6. □

Next, we show that a Y-vertex V cannot have more than a single edge, i.e. $r_V \leq 1$. (see Section 5.2.3 for definition of r_V)

Lemma 7. *Suppose that the curve class of a Y-vertex V is 1) a multiple of the fiber B of $Y \to E$, or 2) $B + F$, where F is a fiber of $\mathcal{L}_Y \to Y$. Suppose $r_V > 1$. In case 1), we have $(ev_2)_*[\overline{\mathcal{M}}_V]^{vir} = 0$. In case 2), let $[pt] \in A^2(\mathbb{F}_1)$ be the Poincare dual of a point in $\mathbb{F}_1 \hookrightarrow \mathcal{L}_Y$, then $(ev_2)_*([pt] \cap [\overline{\mathcal{M}}_V]^{vir}) = 0$.*

Proof. When the curve class is $B + F$, recall that we have $n_V = 1$. The virtual dimension of $\overline{\mathcal{M}}_{g,1+r_V} M(\mathcal{L}_Y(\log \mathcal{L}_E), B + F)$ is $3 + r_V$. The relative evaluation map factors through $\overline{\mathcal{M}}_{g,1+r_V}(\mathcal{L}_Y(\log \mathcal{L}_E), B+F) \xrightarrow{ft} \overline{\mathcal{M}}_{g,r_V}(\mathcal{L}_Y(\log \mathcal{L}_E), B+F) \to E \times 0 \hookrightarrow (E \times \mathbb{P}^1)^{r_V}$. The first map forgets the interior marked point and stabilizes. Note that stabilization will not change the incidence of the relative marked points. We consider the cycle $[pt] \cap \overline{\mathcal{M}}_{g,r_V}(\mathcal{L}_Y(\log \mathcal{L}_E), B + F)$ with $[pt] \in A^2(\mathbb{F}_1)$, in order to consider the locus of maps that pass through an interior point constraint. The resulting locus is an r_V-cycle. We show that $(ev_2)_*([pt] \cap \overline{\mathcal{M}}_{g,r_V}(\mathcal{L}_Y(\log \mathcal{L}_E), B + F)) = 0$. Since ev_2 factors through $E \times 0 \hookrightarrow (E \times \mathbb{P}^1)^{r_V}$, it suffices to see that $r_V > \dim E$. But $\dim E = 1$ and $r_V > 1$ by assumption. Thus, we have the desired vanishing.

When the curve class is B, we have $n_V = 0$. The virtual dimension of $\overline{\mathcal{M}}_{g,r_V}(\mathcal{L}_Y(\log \mathcal{L}_E), B)$ is r_V. Similarly, the evaluation map factors through $E \times 0 \hookrightarrow (E \times \mathbb{P}^1)^{r_V}$, and we have $r_V > \dim E$, since we assume $r_V > 1$. □

Finally, we see that Proposition 4.8 of [**Wan**], which shows that the edge weights of Γ must be $w_{e_1} = 1$ and $w_{e_2} = e-1$, generalizes to higher genus. In particular, it disallows the possibility of a graph consisting of a single Y-vertex V with $\beta_V = eB+F$: if $w_{e_1} \geq 2$, then the class β_V must contain a fiber of $Y \to E$. As a result, the evaluation map $\overline{\mathcal{M}}_V \to \mathcal{L}_E$ is constant and factors through $E \times 0$. Thus, when $\dim E = 1$, we have $(ev_2)_*[\overline{\mathcal{M}}_V]^{vir} = 0$, since virdim $\overline{\mathcal{M}}_V > \dim E$ for all $g \geq 0$.

Proof of Theorem 18. When $\dim X = 2$, Lemmas 5, 6, 7 show that Theorem 18 is true for all $g \geq 0$. □

Remark 13. *In* [**Wan**]*, moduli spaces with built-in point constraints are considered, and the decomposition formula of* [**ACGS**] *is used to handle the point constraint defining* $N_{g,1}(Z, \beta + h)$ *by decomposing* $[\overline{\mathcal{M}}(\mathcal{L}_0)]^{vir}$ *into a sum over rigid tropical maps mapping into the tropicalization of* $\mathcal{L}_0$*. We do not point-constrain our moduli spaces; instead we impose the appropriate incidence conditions when evaluating the invariants arising in the degeneration.*

5.2.5. Evaluation of the invariants associated to Γ

The degeneration formula applied to the graphs Γ in Theorem 18 gives us,

$$[\overline{\mathcal{M}}(\mathcal{L}_0)]^{vir} = \sum_{g_A+g_B+g_C=g} (e-1)F_*\Phi^*\Delta^!\Big([\overline{\mathcal{M}}_{g_A,2}(\mathcal{L}_X(\log \mathcal{L}_E), \beta)]^{vir} \times$$
$$[\overline{\mathcal{M}}_{g_B,1}(\mathcal{L}_Y(\log \mathcal{L}_E), (e-1)B)]^{vir} \times [\overline{\mathcal{M}}_{g_C,2}(\mathcal{L}_Y(\log \mathcal{L}_E), B+F)]^{vir}\Big)$$

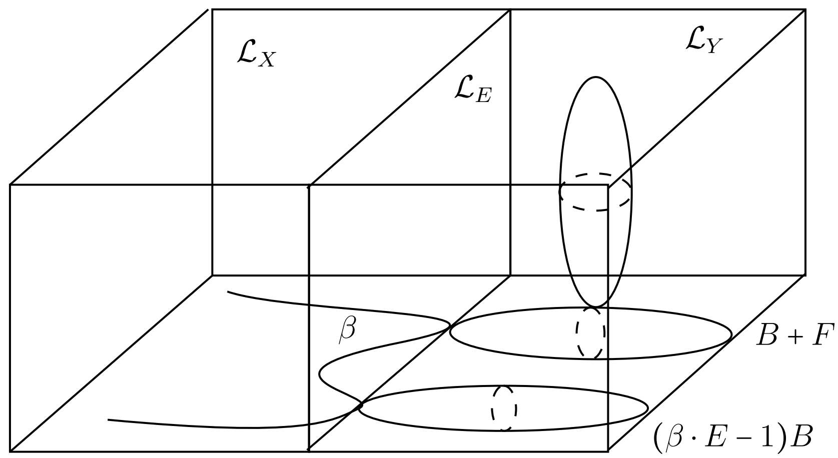


Figure 5.2. The curve classes for the degeneration formula in the central fiber $\mathcal{L}_0$ that is the union of two spaces $\mathcal{L}_X$ and $\mathcal{L}_Y$ intersecting transversely along $\mathcal{L}_E$. These curves are represented by the graph in Theorem 18.

(see Section 5.2.3 for definition of F, Φ, and Δ). The curve invariants on the right hand side are shown in Figure 5.2. We evaluate the right hand side by evaluating the degree of each virtual class appearing in the product.

Let B be a fiber of $Y \to E$ and F be a fiber of $\mathcal{L}_Y \to Y$. The restriction of $\mathcal{L}_Y$ over B is the first Hirzebruch surface $\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$, since $B \cdot (-E_\infty) = -1$. We present $H_2(\mathbb{F}_1, \mathbb{Z})$ in the basis,

$$\mathbb{Z}[B,F]/\langle B^2 = -1, B\cdot F = 1, F\cdot F = 0\rangle \tag{5.1}$$

Effective curve classes $nB + mF$ satisfy $m-n, n \geq 0$. The toric boundary of $\mathbb{F}_1$ is the anti-canonical class $2F + B + \pi^*H$, where π^*H is the pullback of the hyperplane class $H \in H_2(\mathbb{P}^2, \mathbb{Z})$ under the toric blow up $\pi : \mathbb{F}_1 \to \mathbb{P}^2$. We have $\pi^*H = B + F$ is the hyperplane class satisfying $\pi^*H \cdot \pi^*H = 1$.

5.2.5.1. Invariants of Vertex V_1. For vertex V_1, we have $n_{V_1} = 0$ and $r_{V_1} = 2$. The moduli space associated to this vertex is $\overline{\mathcal{M}}_{V_1} := \overline{\mathcal{M}}_{g_{V_1},2}(\mathcal{L}_X(\log \mathcal{L}_E), \beta)$, or genus g_{V_1}, basic stable log maps with two relative contact points to $\mathcal{L}_X(\log \mathcal{L}_E)$ in curve class β, where the first contact point is fixed and has tangency order 1, and the second contact point has tangency order $e-1$. As virtual classes are compatible under base change, we evaluate the degree by Gysin restriction of the evaluation map to $E \times 0$. The moduli space we are then interested in is $\overline{\mathcal{M}}_{g_{V_1},2}(X(\log E), \beta)_{(1,e-1)}$, which has virtual dimension $g+1$. Since the normal bundle $N_{X\times 0/X\times\mathbb{P}^1}|_{X\times 0}$ is isomorphic to $\mathcal{O}_X$, we have the relation of virtual classes $[\overline{\mathcal{M}}(\mathcal{L}_X(\log \mathcal{L}_E), \beta)]^{vir} = (-1)^g \lambda_g \cap [\overline{\mathcal{M}}(X(\log E), \beta)]^{vir}$. The invariant associated to vertex V_1 is therefore,

$$R_{g_{V_1},(1,e-1)}(X(\log E), \beta) := \int_{[\overline{\mathcal{M}}_{g_{V_1},2}(X(\log E),\beta)_{(1,e-1)}]^{vir}} (-1)^{g_{V_1}} \lambda_{g_{V_1}} ev^*([pt]) \tag{5.2}$$

where $[pt] \in A^1(E)$, which is the same two-pointed invariant as in Chapter 4. These invariants can be computed by the tropical/holomorphic correspondence in [**Gra**] or by wall crossing in the quantized scattering diagram of $X(\log E)$.

5.2.5.2. Invariants of Vertex V_2. By Lemma 5.4 of [**vGGR**], we have that $n_{V_2} = 0$ and $r_{V_2} = 1$. The moduli space associated to this vertex is $\overline{\mathcal{M}}_{V_2} := \overline{\mathcal{M}}_{g_{V_2},1}(\mathcal{L}_Y(\log \mathcal{L}_E), (e-1)B)$, or genus g_{V_2}, basic stable log maps to $\mathcal{L}_Y(\log \mathcal{L}_D)$ in curve class $(e-1)B$ with one relative contact point of maximal tangency $(e-1)$. Confining the evaluation map to lie in E, we are interested in the moduli space $\overline{\mathcal{M}}_{g_{V_2},1}(\mathbb{F}_1(\log F), (e-1)B)$, which has virtual dimension g_{V_2}. The obstruction theories defining $[\overline{\mathcal{M}}(\mathcal{L}_Y, (e-1)B)]^{vir}$ and $[\overline{\mathcal{M}}(\mathbb{F}_1, (e-1)B)]^{vir}$ differ by $e(R^1\pi_* f^* N_{\mathbb{F}_1/\mathcal{L}_Y})$. Since the normal bundle of $\mathbb{F}_1 \hookrightarrow \mathcal{L}_Y$ is trivial, the obstruction theories differ by capping with $\lambda_g = e(R^1\pi_* f^* \mathcal{O}_{\mathbb{F}_1})$. The invariant associated to vertex V_2 is therefore,

$$R_{g_{V_2},(e-1)}(\mathbb{F}_1(\log F),(e-1)B) := \int_{[\overline{\mathcal{M}}_{g_{V_2},1}(\mathbb{F}_1(\log F),(e-1)B)]^{vir}} (-1)^{g_{V_2}} \lambda_{g_{V_2}} \tag{5.3}$$

which is the same maximal tangency invariant as in Chapter 4. Since $B \cong \mathbb{P}^1$ is rigid with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, any stable map factors through B. The invariants we want to compute are the local relative invariants of $\mathbb{P}^1$ of [**BP05**]. The obstruction theories defining $[\overline{\mathcal{M}}(\mathbb{F}_1,(e-1)B)]^{vir}$ and $[\overline{\mathcal{M}}(\mathbb{P}^1,(e-1)[\mathbb{P}^1])]^{vir}$ differ by $e(R^1\pi_* f^* N_{B/\mathbb{F}_1})$. As a result, the right hand side of $R_{g_{V_2},(e-1)}(\mathbb{F}_1(\log F),(e-1)B)$ is the invariant,

$$\int_{[\overline{\mathcal{M}}_{g_{V_2},1}(\mathbb{P}^1(\log\infty),(e-1)[\mathbb{P}^1]]^{vir})} e(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)))$$

These invariants were computed in [**BP05**], Theorem 5.1 or [**Bou6**], Lemma 5.9. It is the coefficient of $\hbar^{2g_{V_2}}$ in the expression,

$$\frac{(-1)^e}{(e-1)} \frac{i\hbar}{\mathbf{q}^{(e-1)i\hbar/2} - \mathbf{q}^{-(e-1)i\hbar/2}}$$

with $\mathbf{q} = e^{i\hbar}$. It is $\frac{\hbar}{2} \csc \frac{(e-1)\hbar}{2}$, and the first few terms are $\frac{(-1)^e}{(e-1)^2} + \frac{(-1)^e}{24}\hbar^2 + \frac{7(-1)^e(e-1)^2}{5760}\hbar^4 + \ldots$

5.2.5.3. Invariants of Vertex V_3. Vertex V_3 contains the interior marked point, and we have $n_{V_3} = r_{V_3} = 1$. The moduli space associated to this vertex is $\overline{\mathcal{M}}_{V_3} := \overline{\mathcal{M}}_{g_{V_3},2}(\mathcal{L}_Y(\log \mathcal{L}_E), B+F)$, or genus g_C, basic stable log maps to $\mathcal{L}_Y(\log \mathcal{L}_E)$ in curve class $B+F$ (see Equation 5.1) with one fixed interior point and one relative contact point with tangency order 1. It has virtual dimension 4. After Gysin restriction to a point on E, the moduli space is $\overline{\mathcal{M}}_{g_{V_3},2}(\mathbb{F}_1(\log F), B+F)$, which has virtual dimension $g_{V_3}+3$. Notice that the curve class $B+F$ is the hyperplane class π^*H, which has generic contact order 1 with the log structure F.

As maps in $\overline{\mathcal{M}}_{V_3}$ and $\overline{\mathcal{M}}_{V_1}$ are required to map to the same point in $E\times 0$, we define the invariant associated to $\overline{\mathcal{M}}_{V_3}$ by fixing the relative contact point in addition to the fixed interior point,

$$R_{g_{V_3},2}(\mathbb{F}_1(\log F), B+F) := \int_{[\overline{\mathcal{M}}_{g_{V_3},2}(\mathbb{F}_1(\log F),B+F)]^{vir}} (-1)^{g_{V_3}}\lambda_{g_{V_3}} ev_1^*([pt_1])ev_2^*([pt_2]) \tag{5.4}$$

where $[pt_1]\in A^2(\mathbb{F}_1)$ and $[pt_2]\in A^1(F)$. Let $\gamma := (-1)^{g_{V_3}}\lambda_{g_{V_3}} ev_1^*([pt_1])ev_2^*([pt_2]) \in A^{g_{V_3}+3}(\overline{\mathcal{M}}_{V_3})$. In genus 0, this invariant is the number of lines through two points, or 1. In Section A.3 of the Appendix, we evaluate the genus 1 invariant to be $\frac{-1}{24}$. In arbitrary genus g, work in progress predicts that the invariant $R_{g,2}(\mathbb{F}_1(\log F), B+F)$ is given by the $\hbar^{2g}$-coefficient of the expression $(-i)(\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}})$, which is determined by $\mathbf{q}$-refined tropical curve counting [**Bou2**].

Remark 14. *For the moduli space $\overline{\mathcal{M}}_{V_3}$, we have a non-toric divisorial log structure given by the fiber F of $\mathbb{F}_1$. Most of the literature in the log Gromov-Witten theory of toric surfaces has the toric boundary as divisorial log structure* [**Bou2**], [**Bou6**], [**MR**]. *There have been results with a non-toric divisorial log structure. Cavalieri, Johnson, Markwig, and Ranganathan use floor diagrams to relate descendant log Gromov-Witten theory of Hirzebruch surfaces to bosonic Fock space, in which tangency orders of stable log maps are mapped to basis vectors of Fock space.*

By evaluating the degree of $\overline{\mathcal{M}}_V$ for each vertex $V \in V(\Gamma)$, we arrive at the following expression for $N_{g,1}(Z,\beta+h)$.

Proposition 3. *We have,*

$$N_{g,1}(Z,\beta+h) = \sum_{\substack{\Gamma\in\Gamma(g,n,\beta),\\ g=g_{V_1}+g_{V_3}+g_{V_3}}} \Big[(\beta\cdot E-1)R_{g_{V_1},(1,\beta\cdot E-1)}(X(\log E),\beta)\cdot$$

$$R_{g_{V_2},(\beta\cdot E-1)}(\mathbb{F}_1(\log F),(\beta\cdot E-1)B)R_{g_{V_3},2}(\mathbb{F}_1(\log F),B+F)\Big]$$

where the invariants $R_{g_{V_1},(1,\beta\cdot E-1)}(X(\log E),\beta), R_{g_{V_2},(\beta\cdot E-1)}(\mathbb{F}_1(\log F),(\beta\cdot E-1)B)$*, and* $R_{g_{V_3},2}(\mathbb{F}_1(\log F),B+F)$ *are respectively given in Equations 5.2, 5.3, 5.4.*

Proof. We refer to Sections 5.2.5.1, 5.2.5.2,5.2.5.3 for the description of the invariants. We order the edges of the bipartite graph Γ in 18 such that the edge representing the fixed relative contact point is the top edge. Then, we apply the degeneration formula of [**KLR**]. □

5.3. Obtaining all genus local Gromov-Witten invariants

We prove Theorem 14 which relates invariants $N_{g,1}(Z)$ of the projective compactification to local invariants of the blow up $\hat{X}$. Our main tool is the higher genus log-local principle of [**BFGW**]. We provide explicit formulas in genus 1 and 2.

We introduce formal variables Q and $\hbar$ to keep track of effective curve classes and genus, respectively. Given a curve class $\beta \in H_2^+(X,\mathbb{Z})$ with $e := \beta\cdot E$, define the generating functions,

$$F_{V_2} = \sum_{g_{V_2}\geq 0} R_{g_{V_2},1}(\mathbb{F}_1(\log F),(e-1)B)\hbar^{2g_B}$$

$$F_{V_3} = \sum_{g_{V_3}\geq 0} R_{g_{V_3},2}(\mathbb{F}_1(\log F),B+F)\hbar^{2g_{V_3}}$$

Note that F_{V_2} and F_{V_3} are independent of β.

Proof of Theorem 14. We sum over all genus in Proposition 3 to have,

$$\sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g} = (\beta\cdot E-1)\left(\sum_{g_{V_1}\geq 0} R_{g_{V_1},(1,\beta\cdot E-1)}(X(\log E),\beta)\hbar^{2g_{V_1}}\right)F_{V_2}F_{V_3} \tag{5.5}$$

Summing over all curve classes $\beta\in H_2^+(X,\mathbb{Z})$ and applying Corollary 6.6 of [**GRZ**] to $R_{g_{V_1},2}(X(\log E))$ in Equation 5.5, we have that,

(5.6)

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g}Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\Bigg[(\beta\cdot E-1)\Bigg(\sum_{g_{V_1}\geq 0}\Big[R_{g_{V_1},(\beta\cdot E-1)}(\hat{X},\pi^*\beta-C) - \sum_{i=0}^{g_{V_1}-1} R_{i,(1,\beta\cdot E-1)}(X(\log E),\beta)N(g_{V_1}-i,1)\Big]\hbar^{2g_{V_1}}\Bigg)F_{V_2}F_{V_3}\Bigg]Q^\beta$$

Applying the $g>0$ log-local principle [**BFGW**] (Theorem 25 of the Appendix) to $R_{g_{V_1}}(\hat{X},\pi^*\beta-C)$ in Equation 5.6, it becomes,

(5.7)

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}N_{g,1}(Z,\beta+h)\hbar^{2g}=\sum_{\beta\in H_2^+(X,\mathbb{Z})}\Bigg[(\beta\cdot E-1)\Bigg(\sum_{g_{V_1}\geq 0}\Big[(-1)^{\beta\cdot E}(\beta\cdot E-1)\Big[N_{g_{V_1}}(K_{\hat{X}},\pi^*\beta-C)$$

$$-\sum_{n\geq 0}\sum_{\substack{g_{V_1}=h+g_1+\ldots+g_n,\\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0},\\ \pi^*\beta-C=d_E[E]+\beta_1+\ldots+\beta_n,\\ d_E\geq 0,\beta_j\cdot D>0}}\frac{(-1)^{g_{V_1}-1+(E\cdot E)d_E}(E\cdot E)^m}{m!|Aut(\mathbf{a},g_{V_1})|}N_{h,(\mathbf{a},1^m)}(E,d_E)$$

$$\prod_{j=1}^{n}((-1)^{\beta_j\cdot E}(\beta_j\cdot E)R_{g_j,(\beta_j\cdot E)}(\hat{X},\beta_j)\Big]$$

$$-\sum_{i=0}^{g_{V_1}-1}R_{i,(1,\beta\cdot E-1)}(X(\log E),\beta)N(g_{V_1}-i,1)\Big]\hbar^{2g_{V_1}}\Bigg)F_{V_2}F_{V_3}\Bigg]Q^{\beta}$$

Define Δ^{pl} to be the term in the right hand side of Equation 5.7 given by,

(5.8)

$$\Delta^{pl}:=\sum_{\beta\in H_2^+(X,\mathbb{Z})}\Bigg[(\beta\cdot E-1)\Bigg(\sum_{g_{V_1}\geq 0}\Bigg[(-1)^{\beta\cdot E}(\beta\cdot E-1)\sum_{n\geq 0}\Bigg[\sum_{\substack{g_{V_1}=h+g_1+\ldots+g_n,\\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0},\\ \beta=d_E[E]+\beta_1+\ldots+\beta_n,\\ d_E\geq 0,\beta_j\cdot D>0}}\frac{(-1)^{g_{V_1}-1+(E\cdot E)d_E}(E\cdot E)^m}{m!|Aut(\mathbf{a},g_{V_1})|}$$

$$N_{h,(\mathbf{a},1^m)}(E,d_E)\prod_{j=1}^{n}((-1)^{\beta_j\cdot E}(\beta_j\cdot E)R_{g_j,(\beta_j\cdot E)}(\hat{X},\beta_j)\Bigg]$$

$$+\sum_{i=0}^{g_{V_1}-1}R_{i,(1,\beta\cdot E-1)}(X(\log E),\beta)N(g_{V_1}-i,1)\Bigg]\hbar^{2g_{V_1}}\Bigg)F_{V_2}F_{V_3}\Bigg]Q^{\beta}$$

For $g\geq 0$ and $\beta\in H_2^+(X,\mathbb{Z})$, define $\Delta^{pl}(g,\beta)$ by the expression,

(5.9)
$$\Delta^{pl} = \sum_{g_{V_1}\geq 0} \sum_{\beta\in H_2^+(X,\mathbb{Z})} \Delta(g_{V_1},\beta)^{pl}\hbar^{2g_{V_1}}Q^{\beta}$$

Substituting Δ^{pl} into Equation 5.7 simplifies to,

(5.10)
$$\sum_{\beta\in H_2^+(X,\mathbb{Z})} \sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g} = \sum_{\beta\in H_2^+(X,\mathbb{Z})} \left[(-1)^{\beta\cdot E}(\beta\cdot E-1)^2\left(\sum_{g_{V_1}\geq 0} N_{g_{V_1}}(K_{\hat{X}},\pi^*\beta-C)\hbar^{2g_{V_1}}\right)F_{V_2}F_{V_3}\right]Q^{\beta} - \Delta^{pl}$$

For each $g_{V_1}\geq 0$ and $\beta\in H_2^+(X,\mathbb{Z})$, there exists a constant $c(g_{V_1},\beta)$ that represents the overall contribution of $F_{V_2}F_{V_3}$ to the coefficient of $\hbar^{2g_{V_1}}$. By absorbing $(-1)^{\beta\cdot E}(\beta\cdot E-1)^2$ into $c(g_{V_1},\beta)$ as well, Equation 5.10 becomes,

(5.11)
$$\sum_{\beta\in H_2^+(X,\mathbb{Z})} \sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g} = \sum_{\beta\in H_2^+(X,\mathbb{Z})} \sum_{g_{V_1}\geq 0} \left[c(g_{V_1},\beta)N_{g_{V_1}}(K_{\hat{X}},\pi^*\beta-C)\hbar^{2g_{V_1}}Q^{\beta}\right] - \Delta^{pl}$$

Applying the closed Gopakumar-Vafa formula for toric Calabi-Yau threefolds, Equation 5.11 is equal to,

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})} \sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g} = \sum_{\beta\in H_2^+(X,\mathbb{Z})} \sum_{g_{V_1}\geq 0} \left[c(g_{V_1},\beta)n_{g_{V_1}}(K_{\hat{X}},\pi^*\beta-C)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}}}\right)^{2g_{V_1}-2}Q^{\beta}\right]-\Delta^{pl}$$

Relabelling g_{V_1} by g proves Theorem 14. □

Remark 15. *By the degeneration argument of Corollary 6.6 of* [**GRZ**]*, the maximal tangency invariants* $R_{g,1}(\hat{X})$ *can be expressed in terms of two-pointed invariants* $R_{h,2}(X(\log E))$ *for* $h \leq g$*. Therefore, the discrepancy term* Δ^{pl} *is a function of the stationary invariants* $N_{h,(\mathbf{a},1^m)}(E, d_E)$ *of the elliptic curve and two-pointed log invariants* $R_{h,2}(X(\log E), \beta)$ *for* $h \leq g$*. We refer to the Theorem 25 in the Appendix for the definition of the stationary invariants* $N_{h,(\mathbf{a},1^m)}(E, d_E)$*.*

5.3.1. Formulas in genus 1 and 2

We give an explicit formula for Theorem 14 in genus 1 and a conjectured formula in genus 2. For simplicity, we will at times suppress notation for the log structure or curve class by writing $R_{g,(p,q)}(X(\log E), \beta)$ as $R_{g,(p,q)}(X)$.

Corollary 1 (Theorem 1 in genus 1)**.** *Let* $\beta \in H_2^+(X, \mathbb{Z})$*. In genus 1, we have the equality,*

$$N_{1,1}(Z, \beta + h) = n_1(K_{\hat{X}}, \pi^*\beta - C) - \delta_1(\beta)$$

where $n_1(K_{\hat{X}}, \pi^*\beta - C)$ *is the genus 1, Gopakumar-Vafa invariant of* $K_{\hat{X}}$*, and* $\delta_1(\beta)$ *is expressed in terms of stationary invariants of the elliptic curve defined in Appendix A, Equation 38. Summing over all curve classes, we have,*

$$\sum_{\beta \in H_2^+(X,\mathbb{Z})} N_{1,1}(Z, \beta + h) Q^\beta = \sum_{\beta \in H_2^+(X,\mathbb{Z})} [n_1(K_{\hat{X}}, \pi^*\beta - C) - \delta_1(\beta)] Q^\beta$$

Proof. Let $\beta \in H_2^+(X, \mathbb{Z})$ be an effective curve class, and $e := \beta \cdot E$. The respective genus 1 invariants in F_{V_2} and F_{V_3} are $\frac{(-1)^e}{24}$ and $\frac{-1}{24}$. By Proposition 3, we have that,

$$N_{1,1}(Z,\beta+h) = (e-1)\Bigg[\frac{(-1)^e}{(e-1)^2}R_{1,(1,e-1)}(X(\log E),\beta) + \frac{(-1)^e}{24}R_{0,(1,e-1)}(X(\log E),\beta) + \frac{(-1)^{e+1}}{24(e-1)^2}R_{0,(1,e-1)}(X(\log E),\beta)\Bigg] \tag{5.12}$$

Applying Corollary 6.6 of [**GRZ**] to $R_{1,(1,e-1)}(X)$ on the right hand side of Equation 5.12, we have,

$$N_{1,1}(Z,\beta+h) = \frac{(-1)^e}{(e-1)}R_{1,(e-1)}(\hat{X}) + \left(\frac{(-1)^{e+1}}{24(e-1)} + \frac{(-1)^e(e-1)}{24} + \frac{(-1)^{e+1}}{24(e-1)}\right)R_{0,(1,e-1)}(X) \tag{5.13}$$

The genus 1, log-local principle [**BFGW**] tells us that,

$$R_{1,(e-1)}(\hat{X}) = (-1)^e(e-1)\left[N_1(K_{\hat{X}}) + \frac{(-1)^{e+1}(e-1)}{24}R_{0,(e-1)}(\hat{X}) - \delta_1(\beta)\right]$$

where $\delta_1(\beta)$ is defined in Equation 38 in the Appendix. Applying this to Equation 5.13, we have,

$$N_{1,1}(Z,\beta+h) = N_1(K_{\hat{X}}) + \frac{(-1)^{e+1}(e-1)}{24}R_{0,(e-1)}(\hat{X}) - \delta_1(\beta) + \left(\frac{(-1)^{e+1}}{24(e-1)} + \frac{(-1)^e(e-1)}{24} + \frac{(-1)^{e+1}}{24(e-1)}\right)R_{0,(1,e-1)}(X,\beta) \tag{5.14}$$

The $g = 1$ closed Gopakumar-Vafa formula for Calabi-Yau threefolds for the primitive curve class $\pi^*\beta - C$ states,

$$N_1(K_{\hat{X}}, \pi^*\beta - C) = n_1(K_{\hat{X}}, \pi^*\beta - C) + \frac{1}{12} n_0(K_{\hat{X}}, \pi^*\beta - C) \tag{5.15}$$

We also have by Corollary 6.6 of [**GRZ**] and the $g = 0$ log-local principle [**vGGR**] the first and second equality, respectively,

(5.16)

$$R_{0,(1,e-1)}(X(\log E), \beta) = R_{0,(e-1)}(\hat{X}(\log \pi^* E - C), \pi^*\beta - C) = (-1)^e (e-1) n_0(K_{\hat{X}}, \pi^*\beta - C)$$

Applying both Equations 5.15,5.16 to Equation 5.14, we have,

(5.17)

$$\begin{aligned} N_{1,1}(Z, \beta + h) = n_1(K_{\hat{X}}) + \frac{(-1)^e}{12(e-1)} R_{0,(1,e-1)}(X) + \frac{(-1)^{e+1}(e-1)}{24} R_{0,(e-1)}(\hat{X}) - \delta_1(\beta) \\ + \left(\frac{(-1)^{e+1}}{24(e-1)} + \frac{(-1)^e(e-1)}{24} + \frac{(-1)^{e+1}}{24(e-1)} \right) R_{0,(1,e-1)}(X, \beta) \end{aligned}$$

Using $R_{0,(1,e-1)}(X) = R_{0,(e-1)}(\hat{X})$ ([**GRZ**], Corollary 6.6), the coefficients of $R(X)$ and $R(\hat{X})$ sum to 0 in Equation 5.17. Thus we have the equality,

$$N_{1,1}(Z, \beta + h) = n_1(K_{\hat{X}}, \pi^*\beta - C) - \delta_1(\beta)$$

Summing over all curve classes in X, we have the desired expression. □

Remark 16. *The genus 0 result in* [**Cha**] *proves that* $N_{0,1}(Z,\beta+h) = O_0(K_X,\beta+\beta_0,1)$*, and* [**LLW**] *show that* $O_0(K_X,\beta+\beta_0,1) = N_0(K_{\hat{X}},\pi^*\beta - C)$*. In genus 0, the Gopakumar-Vafa formula tells us that* $N_0(K_{\hat{X}},\pi^*\beta - C) = n_0(K_{\hat{X}},\pi^* - C)$*. Hence, we have*

$$N_{0,1}(Z,\beta+h) = n_0(K_{\hat{X}},\pi^* - C)$$

In Corollary 1, we see that in genus 1, the equality is corrected by stationary invariants of the elliptic curve $\delta_1(\beta)$*.*

We provide a conjectured corollary of Theorem 1 in genus 2, with the assumption that the genus 2 invariant of Vertex V_3 is given by the $\hbar^4$-coefficient of $(-i)(\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}})$ (see Section 5.2.5.3).

Corollary 2 (Theorem 1 in genus 2)**.** *In genus 2, we have,*

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})} N_{2,1}(Z,\beta+h)Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})} \left[(-1)^{\beta\cdot E}(\beta\cdot E-1)^2 n_2(K_{\hat{X}},\pi^*\beta - C) - \Delta^{pl,*}(2,\beta)\right]Q^\beta$$

where $n_2(K_{\hat{X}},\pi^*\beta - C)$ *is the genus 2, Gopakumar-Vafa invariant of* $K_{\hat{X}}$*, and* $\Delta^{pl,*}(2,\beta)$ *is defined in Equation 5.21.*

Proof. Let $\beta \in H_2^+(X,\mathbb{Z})$ be an effective curve class, and $e := \beta\cdot E$. The genus 2 invariant in F_{V_2} is $\frac{7(-1)^e(e-1)^2}{5760}$ and the genus 2 invariant in F_{V_3} is conjectured to be $\frac{1}{1920}$, respectively. By Proposition 3, we have,

$$N_{2,1}(Z,\beta+h) = (e-1)\Bigg(\frac{(-1)^e}{(e-1)^2}R_{2,(1,e-1)}(X(\log E))$$
$$+\left[\frac{(-1)^{e+1}}{24(e-1)^2}+\frac{(-1)^e}{24}\right]R_{1,(1,e-1)}(X(\log E)) \tag{5.18}$$
$$+\left[\frac{(-1)^e}{1920(e-1)^2}+\frac{7(-1)^e(e-1)^2}{5760}+\frac{(-1)^{e+1}}{576}\right]R_{0,(1,e-1)}(X(\log E))\Bigg)$$

Applying Corollary 6.6 in [**GRZ**] to $R_{2,(1,e-1)}(X(\log E))$ in the right hand side of Equation 5.18, it becomes,

(5.19)
$$N_{2,1}(Z,\beta+h) = (e-1)\Bigg(\frac{(-1)^e}{(e-1)^2}\left[R_{2,(e-1)}(\hat{X})-\frac{R_{1,(1,e-1)}(X(\log E,\beta)}{24}-\frac{7R_{0,(1,e-1)}(X(\log E),\beta)}{5760}\right]$$
$$+\left[\frac{(-1)^{e+1}}{24(e-1)^2}+\frac{(-1)^e}{24}\right]R_{1,(1,e-1)}(X(\log E),\beta)$$
$$+\left[\frac{(-1)^e}{1920(e-1)^2}+\frac{7(-1)^e(e-1)^2}{5760}+\frac{(-1)^{e+1}}{576}\right]R_{0,(1,e-1)}(X(\log E),\beta)\Bigg)$$

Applying $g>0$ log-local to the above, we have,

$$N_{2,1}(Z,\beta+h) = (-1)^e(e-1)^2 N_2(K_{\hat{X}},\pi^*\beta-C)-\Delta^{pl}(2,\beta) \tag{5.20}$$

where $\Delta^{pl}(2,\beta)$ is defined in Equation 5.9.

Recall that the genus 2, closed Gopakumar-Vafa formula for Calabi-Yau threefolds states that

$$N_2(K_{\hat{X}},\pi^*\beta-C) = n_2(K_{\hat{X}},\pi^*\beta-C)+\frac{1}{240}n_0(K_{\hat{X}},\pi^*\beta-C)$$

and by the genus 0 log-local principle [**vGGR**], we have,

$$n_0(K_{\hat{X}}) = \frac{(-1)^e}{e-1} R_{0,(e-1)}(\hat{X}, \pi^*\beta - C)$$

We define,

$$\Delta^{pl,*}(2,\beta) := \Delta^{pl}(2,\beta) - \frac{(e-1)}{240} R_{0,(e-1)}(\hat{X}, \pi^*\beta - C) \tag{5.21}$$

Thus, we have,

$$N_{2,1}(Z, \beta + h) = (-1)^e (e-1)^2 n_2(K_{\hat{X}}, \pi^*\beta - C) - \Delta^{pl,*}(2,\beta) \tag{5.22}$$

Summing over all curve classes, we have the corollary. □

5.4. Blow up formulas for Gromov-Witten invariants

Blow up formulas in Gromov-Witten theory have been studied by [**Ga**], [**Hu**] in genus 0. In real dimension 6, all genus blow up formulas for descendant invariants appear in [**HHKQ**]. In logarithmic Gromov-Witten theory, they appear in the work of [**AW**], which allows for more general birational morphisms.

As an application of Theorem 14, we prove a blow up formula for invariants of projective bundles in all genus. Our result differs from previous results, as it allows for higher genus invariants with only a single point constraint and curve classes that are strict transforms. Our main tool is flop invariance of Gromov-Witten invariants of threefolds. We give an explicit formula in genus 1 and a conjectured formula in genus 2 in Corollary 3.

5.4.1. The spaces involved

Recall that we have a log Calabi-Yau surface X with a smooth elliptic curve E, and $Z := \mathbb{P}(K_X \oplus \mathcal{O}_X)$ is the projective compactification of its canonical bundle. There are two distinguished sections $E_0, E_\infty \subset Z$, both isomorphic to X, that correspond to the summands $\mathbb{P}(0 \oplus \mathcal{O}_X)$ and $\mathbb{P}(K_X \oplus 0)$, respectively. We defined the invariants $N_{g,1}(Z)$ associated to the moduli space $\overline{\mathcal{M}}_{g,1}(Z, \beta + h)$ in the above sections.

Let $\pi : \hat{X} \to X$ be the blow up at a single point of X, with exceptional curve C. Define $\hat{Z} := \mathbb{P}(K_{\hat{X}} \oplus \mathcal{O}_{\hat{X}})$. Let $p \in E_\infty$, and $L \cong \mathbb{P}^1 \subset Z$ be the unique fiber passing through p. Define $W := Bl_p Z$, or the blow up at p of Z, with $\pi_1 : W \to Z$. Let $\tilde{L}$ be the strict transform of L under π_1. It is a smooth rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. We call such a curve a $(-1,-1)$-curve.

5.4.2. The invariants

We relate one-pointed Gromov-Witten invariants $N_{g,1}(Z)$ to unmarked invariants of W via the intermediate space $\hat{Z}$. To do this, we define some additional invariants. Let $\overline{\mathcal{M}}_{g,0}(K_{\hat{X}}, \pi^*\beta - C)$ be the moduli space of genus g, unmarked maps to $K_{\hat{X}}$ in the curve class $\pi^*\beta - C$. Since $K_{\hat{X}}$ is Calabi-Yau and of dimension 3, the virtual dimension is 0 by Riemann-Roch. We define,

$$N_{g,0}(K_{\hat{X}}, \pi^*\beta - C) := \int_{[\overline{\mathcal{M}}_{g,0}(K_{\hat{X}}, \pi^*\beta - C)]^{vir}} 1$$

Let $\overline{\mathcal{M}}_{g,0}(W, \beta + \tilde{L})$ be the moduli space of genus g, unmarked maps to W in the curve class $\beta + \tilde{L}$. Since $c_1(TW)(\beta) = c_1(TW)(\tilde{L}) = 0$, its virtual dimension is 0. We define,

$$N_{g,0}(W, \beta + \tilde{L}) := \int_{[\overline{\mathcal{M}}_{g,0}(W,\beta+\tilde{L})]^{vir}} 1$$

We will at times write $N_{g,0}(K_{\hat{X}}, \pi^*\beta - C)$ and $N_{g,0}(W, \beta + \tilde{L})$ as $\langle 1 \rangle^{\hat{X}}_{g,0,\pi^*\beta-C}$ and $\langle 1 \rangle^{W}_{g,0,\beta+\tilde{L}}$ respectively.

5.4.3. Flop Invariance

Invariance of Gromov-Witten invariants of threefolds under birational transformations has garnered considerable interest. In particular, flops are birational transformations that are compositions of blow ups and blow downs along a $(-1,-1)$-curve that have been studied in Gromov-Witten theory. Li and Ruan proved that Gromov-Witten invariants behave functorially with respect to flops [**LR**]. Their result has been used to prove genus 0 open-closed equalities for toric Calabi-Yau threefolds [**LLW**]. It has also motivated wide ranging results in the topological vertex [**KM**], the Crepant Resolution conjecture [**BG**], Gopakumar-Vafa/Stable Pair correspondences [**MT**], and Donaldson-Thomas theory [**HL**]. Flop invariance of Gromov-Witten invariants states that,

Theorem 20. *([**LR**]) For a simple flop $\varphi : X \dashrightarrow Y$ between threefolds, if β is not a multiple of an exceptional curve, i.e. a rational curve with self-intersection -1, then we have the following equality in all genus,*

$$\langle \varphi^*\gamma_1, \ldots, \varphi^*\gamma_n \rangle^X_{g,n,\beta} = \langle \gamma_1, \ldots, \gamma_n \rangle^Y_{g,n,\varphi(\beta)}$$

Using the above theorem, we have the following lemma equating the invariants of W and $\hat{Z}$.

Lemma 8. *For all $g \geq 0$, we have the equality,*

$$N_{g,0}(W, \beta + \tilde{L}) = N_{g,0}(\hat{Z}, \pi^*\beta - C)$$

Proof. There exists a flop $\varphi : W \dashrightarrow \hat{Z}$ along the smooth $(-1,-1)$-curve $\tilde{L}$ such that $\varphi(\tilde{L}) = -C$ (see [**LLW**], Proposition 3.1 for more details). By Theorem 20, we have the desired equality. □

5.4.3.1. Description of the flop for $\mathbb{P}^2$. When the underlying surface is $\mathbb{P}^2$, we describe the flop between Bl_pZ and $\mathbb{P}(K_{\mathbb{F}_1} \oplus \mathcal{O}_{\mathbb{F}_1})$ from their toric fans. Recall that a toric variety is Calabi-Yau, if there exists a vector ν such that $(\nu, v_i) = 1$ for all primitive vectors v_i of its 1-dimensional cones. This implies that toric Calabi-Yau varieties are non-compact, since their fans are not complete.

Consider the fan Σ of local $\mathbb{P}^2$. It is the cone over the convex hull of the fan of $\mathbb{P}^2$. Its one dimensional cones $\Sigma(1)$ are given by the vectors $\{(0,0,1),(1,0,1),(0,1,1),(-1,-1,1)\}$, and its three dimensional cones are the cones over the three smaller triangles in the plane $z = 1$. The fan of local $\mathbb{F}_1$ is similarly defined as the cone over the convex hull of the fan of $\mathbb{F}_1$.

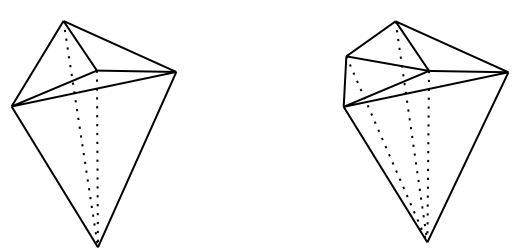


Figure 5.3. The fans of $K_{\mathbb{P}^2}$ and $K_{\mathbb{F}_1}$ respectively on the left and right.

Adding the ray generated by $(0,0,-1)$ to Σ gives the fan (and after completing to a convex fan) of the projective compactification $Z = \mathbb{P}(K_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$, where $(0,0,-1)$ corresponds to the divisor at infinity $E_\infty \subset Z$. The resulting fan is shown in Figure 5.4. Blowing up a toric fixed point $p \in E_\infty$ corresponds to adding the ray $(0,-1,1)$ which is the sum of rays $(1,0,1) + (-1,-1,1) + (0,0,-1)$. The resulting space is thus $W = Bl_p Z$.

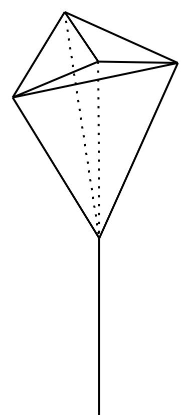


Figure 5.4. The fan of $Z = \mathbb{P}(K_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2})$ that is obtained from that of $K_{\mathbb{P}^2}$ by adding the 1-dimensional cone generated by $(0,0,-1)$ (and completing to a convex fan).

We now describe the flop $W \dashrightarrow \mathbb{P}(K_{\mathbb{F}_1} \oplus \mathcal{O}_{\mathbb{F}_1})$ shown in Figure 5.5. The four rays $(0,0,1), (1,0,1), (0,-1,1)$, and $(-1,-1,1)$ in the fan of W form a quadrilateral in the plane $z = 1$, with a diagonal connecting $(1,0,1)$ and $(-1,-1,1)$. The flop along the $(-1,-1)$-rational curve switches the diagonal to connect $(0,0,1)$ and $(0,-1,1)$. The resulting fan is that of $\mathbb{P}(K_{\mathbb{F}_1} \oplus \mathcal{O}_{\mathbb{F}_1})$.

5.4.4. Proof of Theorem 15

We first have a lemma that shows the Gromov-Witten invariants of $K_{\hat{X}}$ and $\hat{Z}$ in curve classes $\pi^*\beta - C$ are equal.

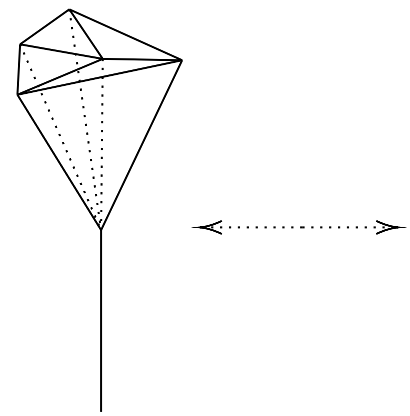


Figure 5.5. The flop between the spaces $\mathbb{P}(K_{\mathbb{F}_1} \oplus \mathcal{O}_{\mathbb{F}_1})$ and $W = Bl_p Z$, whose fans are respectively on the left and right (and completing to convex fans). It is the composition of a blow up and blow down of a smooth rational $(-1,-1)$-curve.

Lemma 9. *For all $g \geq 0$, we have that $N_{g,0}(K_{\hat{X}}, \pi^*\beta - C) = N_{g,0}(\hat{Z}, \pi^*\beta - C)$.*

Proof. Under the $\mathbb{C}^*$-action that scales the $\mathbb{P}^1$-fiber, the fixed point set of $\overline{\mathcal{M}}_{g,0}(\hat{Z}, \pi^*\beta - C)$ is isomorphic to $\overline{\mathcal{M}}_{g,0}(\hat{X}, \pi^*\beta - C)$. By virtual localization, the two invariants are equal (see [**KM**], Proposition 2.2). □

By blowing up, we effectively rid of the point constraint defining $N_{g,1}(Z, \beta + h)$ by considering invariants with one less point constraint to the blown up space $W = Bl_p Z$.

Proof of Theorem 15. Applying the Gopakumar-Vafa formula to the right side of Theorem 14, there exist constants $c(g, \beta) \in \mathbb{Q}$ such that,

(5.23)
$$\sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} N_{g,1}(Z, \beta + h) \hbar^{2g} Q^\beta = \sum_{\beta \in H_2^+(X,\mathbb{Z})} \sum_{g \geq 0} c(g, \beta) N_{g,0}(K_{\hat{X}}, \pi^*\beta - C) \hbar^{2g} Q^\beta - \Delta^{pl}$$

By Lemma 9, Equation 5.23 becomes,

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g}Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} c(g,\beta)N_{g,0}(\hat{Z},\pi^*\beta - C)\hbar^{2g}Q^\beta - \Delta^{pl} \tag{5.24}$$

By Lemma 8, Equation 5.24 becomes,

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} N_{g,1}(Z,\beta+h)\hbar^{2g}Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} c(g,\beta)N_{g,0}(W,\beta+\tilde{L})\hbar^{2g}Q^\beta - \Delta^{pl} \tag{5.25}$$

and we have shown the desired conclusion. □

5.4.5. Formulas in Genus 1 and 2

We give an explicit blow up formula in genus 1 and conjectured formula in genus 2.

Corollary 3 (Theorem 2 in genus 1 and 2)**.** *In genus 1, Theorem 15 reduces to,*

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})} N_{1,1}(Z,\beta+h)Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\left[N_{1,0}(W,\beta+\tilde{L}) - \frac{1}{12}N_{0,0}(W,\beta+\tilde{L}) - \delta_1(\beta)\right]Q^\beta$$

and in genus 2, we are conjectured to have,

$$\begin{aligned}\sum_{\beta\in H_2^+(X,\mathbb{Z})} N_{2,1}(Z,\beta+h)Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}(-1)^{\beta\cdot E}(\beta\cdot E-1)^2\Big[N_{2,0}(W,\beta+\tilde{L}) - \frac{1}{240}N_{0,0}(W,\beta+\tilde{L})\\ - \Delta^{pl,*}(2,\beta)\Big]Q^\beta\end{aligned}$$

with $\delta_1(\beta)$ defined in Equation 38, and $\Delta^{pl,*}(2,\beta)$ defined in Equation 5.21.

Proof. By Corollary 1, the genus 1 Gopakumar-Vafa formula for threefolds, and Lemmas 9, 8 respectively, we have that,

$$
\begin{aligned}
N_{1,1}(Z,\beta+h) &= n_{1,0}(K_{\hat{X}}) - \delta_1(\beta)\\
&= N_{1,0}(K_{\hat{X}},\pi^*\beta - C) - \frac{1}{12}N_{0,0}(K_{\hat{X}},\pi^*\beta - C) - \delta_1(\beta)\\
&= N_{1,0}(\hat{Z},\pi^*\beta - C) - \frac{1}{12}N_{0,0}(\hat{Z},\pi^*\beta - C) - \delta_1(\beta)\\
&= N_{1,0}(W,\beta+\tilde{L}) - \frac{1}{12}N_{0,0}(W,\beta+\tilde{L}) - \delta_1(\beta)
\end{aligned}
$$

Summing over all curve classes, we have the desired expression. The conjectured case for genus 2 is similar. □

Remark 17. *In* [**HHKQ**]*, Theorem 1.1, they use the degeneration formula to prove a blow up formula for descendant invariants of threefolds. Theorem 15 extends their result by covering cases when invariants have only a single point constraint.*

5.5. Open-closed conjectures for projective bundles

Applying Conjecture 9 to Theorem 14, Conjecture 4 follows. We remark that in genus 0, Conjecture 4 was proven in [**Cha**]. In genus 1 and 2, Conjecture 4 takes the following form,

Conjecture 5 (Conjecture 2 in genus 1 and 2)**.** *Let $\beta \in H_2^+(X,\mathbb{Z})$. In genus 1, we have,*

$$N_{1,1}(Z,\beta+h) = n_1^{open}(K_X,\beta+\beta_0,1) - \delta_1(\beta)$$

where $\delta_1(\beta)$ is defined in Equation 38.

In genus 2, we have,

$$N_{2,1}(Z,\beta+h) = (-1)^{\beta\cdot E+1}(\beta\cdot E-1)^2 n_2^{open}(K_X,\beta+\beta_0,1) - \Delta^{pl,*}(2,\beta)$$

where $\Delta^{pl,*}(2,\beta)$ *is defined in Equation 5.21.*

Summing over all curve classes, we have,

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})} N_{1,1}(Z,\beta+h)Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})} [n_1^{open}(K_X,\beta+\beta_0,1) - \delta_1(\beta)]Q^\beta$$

and

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})} N_{2,1}(Z,\beta+h)Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})} \left[(-1)^{\beta\cdot E+1}(\beta\cdot E-1)^2 n_2^{open}(K_X,\beta+\beta_0,1)Q^\beta - \Delta^{pl,*}(2,\beta)\right]Q^\beta$$

We now prove Theorem 16,

Proof of Theorem 16. This follows by Theorem 24 applied to Theorem 14. □

CHAPTER 6

Open-log conjecture for log Calabi-Yau surfaces with smooth anti-canonical divisor, with results for $\mathbb{P}^2$

In this chapter, we conjecture an all genus correspondence between two-pointed logarithmic invariants of $X(\log E)$ with winding 1 and framing 0 open invariants of an outer Aganagic-Vafa brane in the canonical bundle K_X, and show that it is equivalent to a conjecture relating open and closed BPS invariants from the topological vertex. When $X = \mathbb{P}^2$, we provide a proof in low degrees and all genus. Our methods rely on the scattering diagrams of Gross-Siebert. We provide explicit formulas of the open-log correspondence in genus 1 and 2. We provide computational validity for the correspondence in various cases. We discuss an application to quantum theta functions and wavefunctions.

6.1. Introduction

Recall that X is a toric Fano surface with a smooth anticanonical divisor E, and $\pi : \hat{X} \to X$ is blow up of X at a point. Let $\beta \in H_2^+(X, \mathbb{Z})$ be an effective curve class, and $e := \beta \cdot E$. From Chapter 4, recall that we have the genus g, two-pointed log invariant $R_{g,(1,e-1)}(X(\log E), \beta)$ in class β with one fixed contact point of tangency order 1 and another contact point with order $e - 1$ with E, and the the genus g, winding 1, open Gromov-Witten invariant $O_g(K_X, \beta + \beta_0, 1)$ of the canonical bundle K_X in class $\beta + \beta_0$ with boundary on a a single Aganagic-Vafa brane, with its corresponding open BPS invariant $n_g^{open}(K_X, \beta + \beta_0, 1)$.

6.1.1. Plan

In Section 6.2, we first provide a review of the genus 0 open-log correspondence established by [**GRZ**], in order to use those techniques in higher genus. Then, we establish a correspondence between $\mathbf{q}$-refined tropical curves and higher genus local Gromov-Witten invariants of $\hat{X}$ in Section 6.3,

Theorem 21 (Theorem 3)**.** *Let (X,E) be log Calabi-Yau surface X with smooth anticanonical divisor E. Then, we have,*

$$\sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0} R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)\hbar^{2g}Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\left[(-1)^{\beta\cdot E+g-1}n_g(K_{\hat{X}},\pi^*\beta-C)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}}}\right)^{2g-2}Q^\beta\right]$$
$$-\Delta^{ol}$$

where $R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)$ is the genus g, two-legged, $\mathbf{q}$-refined tropical curve count in the scattering diagram associated to (X,E) (see Chapter 3 for definition of these invariants), and $n_g(K_{\hat{X}},\pi^*\beta-C)$ is the genus g, Gopakumar-Vafa invariant of $K_{\hat{X}}$ in class $\pi^*\beta-C$, Δ^{ol} is a discrepancy term defined in Equation 6.10 that is a function of the stationary Gromov-Witten theory of the elliptic curve and two pointed log invariants of $X(\log E)$, and $\mathbf{q}=e^{i\hbar}$.

Applying Conjecture 9 in Chapter 7 to Theorem 21, we conjecture an open-log correspondence relating genus g, winding 1, framing 0, open Gromov-Witten invariants of an outer AV-brane in K_X to 2-pointed logarithmic Gromov-Witten invariants of $X(\log E)$ when E is smooth,

Conjecture 6 (Open-log conjecture for smooth divisor, Conjecture 1)**.** *Let* (X, E) *and* Δ^{ol} *be as in Theorem 21. Furthermore, assume that* X *is toric, and* $\pi : \hat{X} \to X$ *is a toric blow up. Then, we conjecture the following correspondence,*

$$\sum_{\substack{\beta \in H_2^+(X,\mathbb{Z}),\\ g \geq 0}} (\beta \cdot E - 1) R_{g,(\beta \cdot E-1,1)}(X(\log E), \beta) \hbar^{2g} Q^\beta = \sum_{\substack{\beta \in H_2^+(X,\mathbb{Z}),\\ g \geq 0}} \left[\frac{(-1)^{g+1}}{(\beta \cdot E - 1)} n_g^{open}(K_X, \beta + \beta_0, 1) \left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}}} \right)^{2g-2} Q^\beta \right] - \Delta^{ol}$$

where $R_{g,(\beta \cdot E-1,1)}(X(\log E), \beta)$ *are two-pointed log invariants of* $X(\log E)$ *with* λ_g*-insertion in class* β*, and* $n_g(K_X, \beta + \beta_0, 1)$ *is the genus* g*, winding 1, framing 0, open BPS invariant of a single outer AV-brane* L *in* K_X*, and* $\mathbf{q} = e^{i\hbar}$*.*

We refer to Chapter 4 and 3 for more detailed definitions of the above invariants.

After using the results in Chapter 7, we establish the following theorem from Conjecture 6,

Theorem 22 (Theorem 4)**.** *Let* $X = \mathbb{P}^2$ *and* $H \in H_2(\mathbb{P}^2, \mathbb{Z})$ *the hyperplane class. Then Conjecture 6 holds in curve classes* $\beta = dH$ *for* $d = 1, 2, 3, 4$ *and all genus.*

Consequently, Theorem 22 tells us that genus g, open Gromov-Witten invariants O_g of a toric Calabi-Yau threefold K_X can be expressed in terms of genus h, 2-pointed log invariants $R_{h,(\beta \cdot E-1),1}(X(\log E))$ and the stationary Gromov-Witten theory of the elliptic curve E, the former of which is computed in the Topological Vertex [**AKMV**], and the latter appears in the scattering diagrams of Gross-Siebert mirror symmetry.

In Section 6.4, we give explicit formulas in genus 1 and 2. In Section 6.5, we perform computations verifying the correspondence in various cases.

In Section 6.6, we place Theorem 22 in the context of quantum theta functions from upcoming work [**GRZZ**].

6.1.2. Related work: correspondences with open invariants

We mention previous work that establish correspondences between open invariants and other enumerative invariants associated from a log Calabi-Yau surface (X, E). Previous results in the enumerative geometry of (X, E) usually make the distinction of whether E is singular or smooth. We say that a log Calabi-Yau surface has *maximal boundary* if E is singular.

6.1.2.1. Open-log. The heuristic for open-log correspondences is the following: curves with boundary in winding w can be "capped off" to obtain a closed curve that intersects the divisor at infinity with tangency order w. Logarithmic Gromov-Witten invariants with λ_g-insertion in some sense should be viewed as a definition for higher genus open invariants.

Li and Song showed that their definition of open invariants using stable relative maps in the algebraic category recovers the Ooguri-Vafa multiple cover formula for disc invariants [**LS**].

For log Calabi-Yau surfaces with singular anticanonical divisor, Bousseau, Brini, and van Garrel showed that open invariants are related to four other enumerative theories associated to (X, D), [**BBvG**]. They use the technology of quantum scattering diagrams to conjecture and prove an all genus open-log correspondence for singular divisor. They

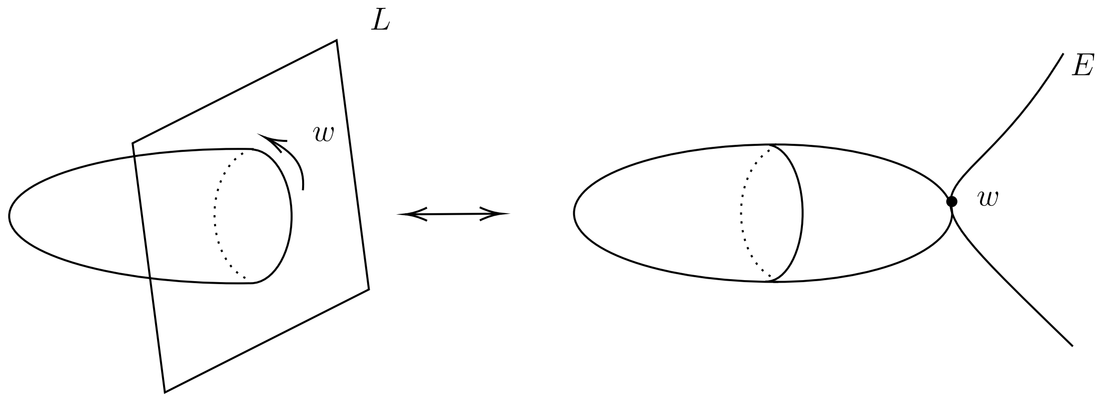


Figure 6.1. Heuristic picture of a holomorphic disc with winding $w \in H_1(L,\mathbb{Z}) \cong \mathbb{Z}$ capped off to create a closed curve intersecting the anti-canonical divisor with tangency order w.

also show that open invariants are related to Donaldson-Thomas invariants of a quiver that can be constructed from the intersection numbers of the curve class with D. Brini and Schuler extend the log-open correspondence of to *quasi-tame* Looijenga pairs that are in a certain sense deformation equivalent by using quantum scattering [**BS**]. Schuler uses the topological vertex to prove an all genus open-log correspondence when $D = D_1 + D_2$ [**Sch**].

BPS invariants associated to genus 0 log invariants were introduced in [**GPS**], and proven to be integers in [**vGWZ**]. Log BPS numbers with λ_g-insertion were defined in [**Bou6**] and were shown to satisfy Ooguri-Vafa duality in all genus when the boundary is maximal.

In the setting of a smooth divisor, Choi, van Garrel, Katz and Takahashi study BPS invariants of local del Pezzo surfaces using the moduli of 1-dimensional sheaves [**CvGKT2**]. They define $g = 0$, log BPS numbers based on open multiple cover formulas, and prove their integrality [**CvGKT1**]. A log-local principle for BPS invariants is given in [**CvGKT3**].

In the context of the topological vertex, Fang and Liu define open invariants of a toric Calabi-Yau threefold by relative invariants of a partially compactified space. They prove their open invariants are expressed in terms of closed equivariant descendant invariants [**FL**], Proposition 3.4.

6.1.2.2. Open-quiver. Bousseau proves an all genus correspondence between log invariants with λ_g-insertion with refined quiver DT invariants when the divisor D has 2 components [**Bou1**], which is an example of the Gromov-Witten/Kronecker correspondence. In [**Zas**], a conjecture is made equating open BPS invariants of an Aganagic-Vafa brane in $\mathbb{C}^3$ with quiver DT invariants of a loop quiver. In related work, Schrader, Shen, and Zaslow study the open Gromov-Witten invariants of Lagrangians in $\mathbb{C}^3$ with a Legendrian link as boundary, and conjecture that such invariants satisfy Kontsevich-Soibelman integrality [**SSZ**]. In [**Bou6**], quantized scattering diagrams are used to conjecture a log BPS/quiver DT equality.

6.1.2.3. Open-closed. In [**Cha**], a genus 0 equality is established between open invariants of a moment fiber of a toric canonical bundle with local invariants of the toric blow up.

Liu and Yu use the open/relative correspondence established in [**FL**] to relate $g = 0$, open invariants of an AV-brane in a toric Calabi-Yau threefold to maximally tangent, relative invariants of its toric partial compactification $(\hat{Y}, \hat{D})$. The latter are then related to closed invariants of the CY4-fold $\mathcal{O}(-D)$ [**LY**].

In symplectic geometry, Chan, Lau, Leung, and Tseng equate open invariants of toric Kähler manifold with closed invariants of certain $\mathbb{P}^1$-bundles used in the construction of Seidel representations [**CLLT**].

6.2. Review of $g = 0$ open-log correspondence

Gräfnitz, Ruddat, and Zaslow show that the proper Landau-Ginzburg potential associated to (X, E) is equivalent to the open mirror map of an outer Aganagic-Vafa brane of framing 0 in K_X, by establishing a genus 0 equivalence between certain two-pointed log invariants of (X, E) and open invariants of K_X with boundary on an Aganagic-Vafa brane [**GRZ**]. They use the cluster variety structure of the toric degeneration associated to (X, E). We review the methods used in [**GRZ**] in order and give their higher genus analogues to formulate an all genus open-log correspondence in Section 6.3.

6.2.1. From tropical curves to log invariants.

Recall from Chapter 3 that $R^{trop}_{g,(p,q)}(X, \beta, \mathbf{q}))$ is the $\mathbf{q}$-refined count of two-legged tropical curves in X with one unbounded leg of weight p and another unbounded leg of weight q in the scattering diagram of (X, E).

Gräfnitz proves a correspondence theorem for two-pointed log invariants of log Calabi-Yau surfaces by using the decomposition formula of [**ACGS**], and counting tropicalizations of stable log maps in the associated dual intersection complex. His tropical/holomorphic correspondence theorem gives the following equality,

$$R^{trop}_{g,(p,1)}(X, \beta) = pR_{g,(p,1)}(X(\log E), \beta) \tag{6.1}$$

We switch the contact orders of the fixed and varying points of intersection with E of the log invariants by using an identity of [**CC**], namely,

$$R_{g,(p,1)} = \frac{1}{p^2} R_{g,(1,p)} \tag{6.2}$$

6.2.2. Trading a contact point by blowing up

Now take the genus g, maximal tangency, log invariant $R_{g,(p)}(\hat{X}(\log \pi^* E - C), \pi^*\beta - C)$ of the blow up $\hat{X} \to X$ in class $\pi^*\beta - C$. These curves will intersect $\pi^* D - C$ with order $p = \beta \cdot E - 1$. By taking the blow up of the degeneration to the normal cone, we have the following formula relating log invariants of blow up $\hat{X}$ to that of the base X ([**GRZ**], Corollary 6.6),

$$R_{g,(p)}(\hat{X}(\log \pi^* E - C), \pi^*\beta - C) = \sum_{g_0 + g_1 = g} R_{g_0,(1,p)}(X(\log E), \beta) N(g_1, 1) \tag{6.3}$$

where $N(g,1)$ denotes the genus g, maximal tangency log invariants of $\mathcal{O}_{\mathbb{P}^1}(-1)$ with λ_g-insertion, defined in Section 6.2 of [**GRZ**], and computed by Theorem 5.1 in [**BP05**]. In particular, $N(g,p)$ is the coefficient of $\hbar^{2g}$ in the expression $\frac{(-1)^{p+1}}{p} \frac{i\hbar}{\mathbf{q}^{pi\hbar/2} - \mathbf{q}^{-pi\hbar/2}} = \frac{(-1)^{p+1}\hbar}{2p} \csc(\frac{p\hbar}{2})$, where $\mathbf{q} = e^{i\hbar}$. We have that $N(0,1) = 1$ and $N(1,1) = \frac{1}{24}$.

Note that in $g = 0$, the formula states that,

$$R_{0,(p)}(\hat{X}, \pi^*\beta - C) = R_{0,(1,p)}(X, \beta)$$

6.2.3. $g = 0$ log-local principle

The log-local principle of [**vGGR**] relates maximal tangency log invariants of $X(\log D)$ to local invariants of K_X. Suppose that β is a nef curve class, and let $d := \beta \cdot D > 0$. Let

$\overline{\mathcal{M}}_{0,1}(X(\log D),\beta)$ be the moduli space of genus 0, 1-marked, maximal tangency basic stable log maps to $X(\log D)$ in curve class β. We have a functor $F:\overline{\mathcal{M}}_{0,1}(X(\log D),\beta)\to\overline{\mathcal{M}}_{0,0}(X,\beta)$ that forgets the log structure and the single marked point of the stable log map. Because β is nef, we have the equality of moduli spaces $\overline{\mathcal{M}}(X,\beta)=\overline{\mathcal{M}}(K_X,\beta)$, and hence we may use F to compare virtual classes. The $g=0$ log-local principle of [**vGGR**] states that,

$$F_*[\overline{\mathcal{M}}_{0,1}(X(\log D)]^{vir}=(-1)^{d+1}d[\overline{\mathcal{M}}_{0,0}(\mathcal{O}_X(-D),\beta)]^{vir}$$

which gives us the equality,

$$R_{0,(\beta\cdot D)}(X(\log D),\beta)=(-1)^{\beta\cdot D+1}(\beta\cdot D)N_{0,0}(\mathcal{O}_X(-D),\beta) \tag{6.4}$$

6.2.4. $g=0$ open-closed equality

In [**LLW**], the authors prove that the genus 0, open invariants of the canonical bundle K_X of a toric surface are equal to genus 0, closed invariants of the canonical bundle $K_{\hat{X}}$ of the blow up, namely

$$O_0(K_X,\beta+\beta_0,1)=N_0(K_{\hat{X}},\pi^*\beta-C) \tag{6.5}$$

Their proof uses the invariance of Gromov-Witten invariants under simple flops [**LR**] as well as the work of [**Cha**], which establishes an equivalence of Kuranishi structures to prove an equivalence between open invariants of a canonical bundle K_X with closed invariants of its projective compactification $\mathbb{P}(K_X\oplus\mathcal{O}_X)$. We shall consider higher genus invariants of $\mathbb{P}(K_X \oplus \mathcal{O}_X)$ in Chapter 5. We remark that [**LLW**] does not use the topological vertex to compute the above invariants.

In conclusion, [**GRZ**] establishes the following equality,

Theorem 23 ([**GRZ**]). *For a toric Fano surface X with smooth anticanonical divisor E, we have the following genus 0 equality,*

$$O_0(K_X, \beta + \beta_0, 1) = (-1)^e (e-1) R_{0,(1,e-1)}(X(\log E), \beta)$$

When $X = \mathbb{P}^2$, these invariants give the coefficients of the open mirror map $M(Q)$ of an outer Aganagic-Vafa brane in framing 0,

$$M(Q) = 1 - 2Q + 5Q^2 - 32Q^3 + 286Q^4 - 3038Q^5 + \dots$$

They can be computed by $R^{trop}_{g,(3d-1,1)}(\mathbb{P}^2, dH)$ in the scattering diagram of $(\mathbb{P}^2, E)$.

Proof. Linking together Equations 6.1-6.5 gives the desired equality. □

Remark 18. *For a toric del Pezzo surface X with smooth anticanonical divisor, van Garrel shows that $g = 0$, log BPS numbers are related to $g = 0$, local BPS invariants by an invertible linear transformation defined by Donaldson-Thomas invariants of a loop quiver* [**vG**]. *In genus 0 and primitive curve class, BPS invariants are equal to ordinary Gromov-Witten invariants. Therefore, combined with the open-closed result of* [**LLW**], *the coefficient $(-1)^e(e-1)$ in Theorem 23 should somehow be a loop quiver Donaldson-Thomas invariant.*

Remark 19. *By* [**vGGR**], [**LLW**] *and the Gromov-Witten/Donaldson-Thomas correspondence* [**MNOP**] *applied to local* $\hat{X}$*, Theorem 23 implies that the open invariants* $O_0(K_X, \beta+\beta_0, 1)$ *can somehow be expressed in terms of Donaldson-Thomas invariants defined by moduli of ideal sheaves, and thus so can the open mirror map* $M(Q)$*. The Gromov-Witten/Donaldson-Thomas correspondence for toric Calabi-Yau threefolds is known to be equivalent to the topological vertex, which suggests a link between the topological vertex and mirror symmetry for toric Calabi-Yau 3-folds.*

6.3. Open-log conjecture and proof of Theorem 4

We formulate Conjecture 6 by extending the techniques in [**GRZ**] to higher genus, and prove it in low degrees and all genus. Equations 6.1-6.3 are already phrased in higher genus. In place of Equation 6.4, we use the higher genus log-local principle of [**BFGW**], which is described in Appendix A. In place of the open-closed result of [**LLW**], we use Conjecture 9 in Chapter 7.

We first prove Theorem 21, relating **q**-refined tropical curves to local Gromov-Witten invariants of the blow up $\hat{X}$.

Proof of Theorem 21. By the tropical/holomorphic correspondence in [**Gra**] or Equation 6.1, we have that,

(6.6)

$$\sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})} R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)\hbar^{2g}Q^{\beta} = \sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})} (\beta\cdot E-1)R_{g,(\beta\cdot E-1,1)}(X(\log E),\beta)\hbar^{2g}Q^{\beta}$$

By the Cadman-Chen formula or Equation 6.2, the right hand side of Equation 6.6 becomes,

(6.7)

$$\sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})} R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)\hbar^{2g}Q^\beta = \sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})}\frac{1}{(\beta\cdot E-1)}R_{g,(1,\beta\cdot E-1)}(X(\log E),\beta)\hbar^{2g}Q^\beta$$

By Corollary 6.6 of [**GRZ**] or Equation 6.3, Equation 6.7 becomes,

(6.8)

$$\sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})} R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)\hbar^{2g}Q^\beta = \sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})}\frac{1}{(\beta\cdot E-1)}[R_{g,(\beta\cdot E-1)}(\hat{X}(\log\pi^*E-C),\pi^*\beta-C)$$
$$-\sum_{i=0}^{g-1}R_{i,(1,\beta\cdot E-1)}(X(\log E),\beta)N(g-i,1)]\hbar^{2g}Q^\beta$$

By the $g>0$ log-local correspondence of [**BFGW**] or Theorem 25, Equation 6.8 becomes,

(6.9)

$$\sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})} R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)\hbar^{2g}Q^\beta = \sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})}\Big[(-1)^{\beta\cdot E}\Big[N_g(K_{\hat{X}},\pi^*\beta-C)$$
$$-\sum_{n\geq 0}\sum_{\substack{g=h+g_1+\ldots+g_n,\\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0},\\ \pi^*\beta-C=d_E[E]+\beta_1+\ldots+\beta_n,\\ d_E\geq 0,\beta_j\cdot D>0}}\frac{(-1)^{g-1+(E\cdot E)d_E}(E\cdot E)^m}{m!|Aut(\mathbf{a},g)|}N_{h,(\mathbf{a},1^m)}(E,d_E)$$
$$\prod_{j=1}^{n}((-1)^{\beta_j\cdot E}(\beta_j\cdot E)R_{g_j,(\beta_j\cdot E)}(\hat{X},\beta_j))\Big]$$
$$-\frac{1}{(\beta\cdot E-1)}\sum_{i=0}^{g-1}R_{i,(1,\beta\cdot E-1)}(X(\log E),\beta)N(g-i,1)\Big]\hbar^{2g}Q^\beta$$

Define Δ^{ol} to be the term,

(6.10)

$$\Delta^{ol} := \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\Big[(-1)^{\beta\cdot E}\Big[\sum_{n\geq 0}\sum_{\substack{g=h+g_1+\ldots+g_n,\\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0},\\ \pi^*\beta-C=d_E[E]+\beta_1+\ldots+\beta_n,\\ d_E\geq 0,\beta_j\cdot D>0}}\frac{(-1)^{g-1+(E\cdot E)d_E}(E\cdot E)^m}{m!|Aut(\mathbf{a},g)|}N_{h,(\mathbf{a},1^m)}(E,d_E)$$

$$\prod_{j=1}^{n}((-1)^{\beta_j\cdot E}(\beta_j\cdot E)R_{g_j,(\beta_j\cdot E)}(\hat{X},\beta_j)\Big]$$

$$-\frac{1}{(\beta\cdot E-1)}\sum_{i=0}^{g-1}R_{i,(1,\beta\cdot E-1)}(X(\log E),\beta)N(g-i,1)\Big]\hbar^{2g}Q^{\beta}$$

For $g\geq 0$ and $\beta\in H_2^+(X,\mathbb{Z})$, define,

(6.11)

$$\Delta^{ol}(g,\beta) := (-1)^{\beta\cdot E}\Big[\sum_{n\geq 0}\sum_{\substack{g=h+g_1+\ldots+g_n,\\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0},\\ \pi^*\beta-C=d_E[E]+\beta_1+\ldots+\beta_n,\\ d_E\geq 0,\beta_j\cdot D>0}}\frac{(-1)^{g-1+(E\cdot E)d_E}(E\cdot E)^m}{m!|Aut(\mathbf{a},g)|}N_{h,(\mathbf{a},1^m)}(E,d_E)$$

$$\prod_{j=1}^{n}((-1)^{\beta_j\cdot E}(\beta_j\cdot E)R_{g_j,(\beta_j\cdot E)}(\hat{X},\beta_j)\Big]$$

$$-\frac{1}{(\beta\cdot E-1)}\sum_{i=0}^{g-1}R_{i,2}(X(\log E),\beta)N(g-i,1)$$

Hence, we have,

$$\Delta^{ol} = \sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})}\Delta^{ol}(g,\beta)\hbar^{2g}Q^{\beta}$$

Therefore, Equation 6.9 becomes,

(6.12)

$$\sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})} R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)\hbar^{2g}Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\left[(-1)^{\beta\cdot E}N_g(K_{\hat{X}},\pi^*\beta - C)\hbar^{2g}Q^\beta\right] - \Delta^{ol}$$

Since $\pi^*\beta - C$ is a primitive curve class, the closed Gopakumar-Vafa formula for Calabi-Yau threefolds takes the form,

$$\sum_{g\geq 0} N_g(K_{\hat{X}},\pi^*\beta - C)\hbar^{2g} = \sum_{g\geq 0} n_g(K_{\hat{X}},\pi^*\beta - C)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}}}\right)^{2g-2}$$

Using this, Equation 6.12 becomes,

(6.13)

$$\sum_{g\geq 0}\sum_{\beta\in H_2^+(X,\mathbb{Z})} R^{trop}_{g,(\beta\cdot E-1,1)}(X,\beta)\hbar^{2g}Q^\beta = \sum_{\beta\in H_2^+(X,\mathbb{Z})}\sum_{g\geq 0}\left[(-1)^{\beta\cdot E}n_g(K_{\hat{X}},\pi^*\beta - C)\left(\frac{i\hbar}{\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}}}\right)^{2g-2} Q^\beta\right] - \Delta^{ol}$$

□

Conjecture 9 in Chapter 7 takes the form,

$$n_g(K_{\hat{X}},\pi^*\beta - C) = (-1)^{g+1}n_g^{open}(K_X,\beta+\beta_0,1) \tag{6.14}$$

Plugging this into Theorem 21 and using [**Gra**], we have Conjecture 6.

We now prove Theorem 22.

Proof of Theorem 22. This follows by applying Theorem 24 applied to Conjecture 6 □

6.4. Genus 1 and 2 open-log conjecture

We give explicit formulas for Conjecture 6 in genus 1 and 2, with their proof in low degrees and all genus by Theorem 22.

6.4.1. Genus 1

Applying Equations 6.1-6.3 in higher genus to $R^{trop}_{1,(e-1,1)}(X,\beta)$, we have,

$$R^{trop}_{1,(e-1,1)}(X,\beta) = \frac{1}{e-1}\left(R_{1,(e-1)}(\hat{X},\pi^*\beta - C) - \frac{R_{0,(1,e-1)}(X,\beta)}{24}\right)$$

We apply the genus 1, log-local principle (Section A.1) to $R_{1,(e-1)}(\hat{X},\pi^*\beta - C)$ to get,

$$N_1(K_{\hat{X}},\pi^*\beta - C) = (-1)^e\left[R^{trop}_{1,(e-1,1)}(X,\beta) + \frac{e^2-2e+2}{24(e-1)}R_{0,(1,e-1)}(X,\beta)\right] + \delta_1(\pi^*\beta - C)$$

where $\delta_1(\pi^*\beta - C)$ is defined in Equation 38. The $g=1$, closed Gopakumar-Vafa formula (Equation 4.4) is,

$$N_1(K_{\hat{X}},\pi^*\beta - C) = n_1(K_{\hat{X}},\pi^*\beta - C) + \frac{1}{12}n_0(K_{\hat{X}},\pi^*\beta - C)$$

Hence we have,

$$\begin{aligned} n_1(K_{\hat{X}},\pi^*\beta - C) + \frac{1}{12}n_0(K_{\hat{X}},\pi^*\beta - C) = (-1)^e\Big[&R^{trop}_{1,(e-1,1)}(X,\beta) \\ &+ \frac{e^2-2e+2}{24(e-1)}R_{0,(1,e-1)}(X,\beta)\Big] + \delta_1(\pi^*\beta - C) \end{aligned}$$

In genus 1, Conjecture 9 takes the form,

$$n_1(K_{\hat{X}}, \pi^*\beta - C) = n_1(K_X, \beta + \beta_0, 1)$$

Together with [**LLW**], we have that,

$$\begin{aligned} n_1(K_X, \beta + \beta_0, 1) - \frac{1}{12} n_0(K_X, \beta + \beta_0, 1) = (-1)^e \Big[& R^{trop}_{1,(e-1,1)}(X, \beta) \\ & + \frac{e^2 - 2e + 2}{24(e-1)} R_{0,(1,e-1)}(X, \beta) \Big] + \delta_1(\pi^*\beta - C) \end{aligned}$$

The $g = 1$, open multiple cover formulas (Equation 4.6) tell us that,

$$\begin{aligned} n_0(K_X, \beta + \beta_0, 1) &= -O_0(K_X, \beta + \beta_0, 1) \\ n_1(K_X, \beta + \beta_0, 1) &= O_1(K_X, \beta + \beta_0, 1) - \frac{1}{24} O_0(K_X, \beta + \beta_0, 1) \end{aligned}$$

which suggests,

$$\begin{aligned} O_1(K_X, \beta + \beta_0, 1) = (-1)^e \left[R^{trop}_{1,(e-1,1)}(X, \beta) + \frac{e^2 - 2e + 2}{24(e-1)} R_{0,(1,e-1)}(X, \beta) \right] \\ - \frac{1}{24} n_0(K_{\hat{X}}, \pi^*\beta - C) + \delta_1(\pi^*\beta - C) \end{aligned}$$

Successively applying [**Gra**], [**CC**], and [**GRZ**], Section 1, Step 5, we arrive at,

Conjecture 7 (Conjecture 1 in genus 1)**.** *We have,*

$$O_1(K_X,\beta+\beta_0,1)=(-1)^e(e-1)R_{1,(e-1,1)}(X(\log E),\beta)+\frac{(-1)^e}{24}(e-1)^3R_{0,(e-1,1)}(X,\beta)$$
$$+\delta_1(\pi^*\beta-C)$$

6.4.2. Genus 2

We specialize Conjecture 6 to genus 2. Applying Equations 6.1-6.3 in genus 2, we have,

$$R^{trop}_{2,(e-1,1)}(X,\beta)=\frac{1}{(e-1)}\left(R_{2,(e-1)}(\hat{X},\pi^*\beta-C)-\frac{R_{1,(1,e-1)}}{24}-\frac{7R_{0,(1,e-1)}}{5760}\right)$$

Applying the $g=2$ log-local principle (Section A.2), we have that

$$R^{trop}_{2,(e-1,1)}(X,\beta)=(-1)^{\beta\cdot E}N_2(K_{\hat{X}},\pi^*\beta-C)-\Delta^{ol}(2,\beta) \tag{6.15}$$

The $g=2$, closed Gopakumar-Vafa formula (Equation 4.4) tells us that,

$$N_2(K_{\hat{X}},\pi^*\beta-C)=n_2(K_{\hat{X}},\pi^*\beta-C)+\frac{1}{240}n_0(K_{\hat{X}},\pi^*\beta-C)$$

and the $g=2$, open-closed BPS conjecture (Conjecture 9) takes the form,

$$n_2(K_{\hat{X}},\pi^*\beta-C)=-n_2^{open}(K_X,\beta+\beta_0,1)$$

Hence, we have

$$R^{trop}_{2,(e-1,1)}(X,\beta) = (-1)^{\beta\cdot E}(-n_2^{open}(K_X,\beta+\beta_0,1) - \frac{1}{240}n_0^{open}(K_X,\beta+\beta_0,1)) - \Delta^{ol}(2,\beta) \tag{6.16}$$

The $g = 2$, winding 1, open multiple cover formulas (Equation 4.6) tell us that,

(6.17)

$$\begin{aligned} -n_2^{open}(K_X,\beta+\beta_0,1) &= O_2(K_X,\beta+\beta_0,1) + \frac{1}{24}n_1(K_X,\beta+\beta_0,1) + \frac{7}{5760}n_0(K_X,\beta+\beta_0,1) \\ &= O_2(K_X,\beta+\beta_0,1) + \frac{1}{24}O_1(K_X,\beta+\beta_0,1) - \frac{3}{5760}O_0(K_X,\beta+\beta_0,1) \end{aligned}$$

We plug in Conjecture 7 to Equation 6.17 to get,

(6.18)

$$\begin{aligned} -n_2^{open}(K_X,\beta+\beta_0,1) = O_2(K_X,\beta+\beta_0,1) \\ + \frac{1}{24}[(-1)^e(e-1)R_{1,(e-1,1)}(X,\beta) + \frac{(-1)^e}{24}(e-1)^3R_{0,(e-1,1)}(X,\beta) \\ + \delta_1(\pi^*\beta - C)] + \frac{3}{5760}(-1)^e(e-1)R_{0,(e-1,1)}(X,\beta) \end{aligned}$$

From [**LLW**] and [**vGGR**], we have the equalities,

$$\frac{-1}{240}n_0^{open}(K_X) = \frac{1}{240}N_0(K_{\hat{X}}) = \frac{(-1)^e(e-1)}{240}R_{0,(e-1,1)}(X) \tag{6.19}$$

Thus, plugging in Equations 6.18 and 6.19 into Equation 6.16 and simplifying, we have that,

Conjecture 8 (Conjecture 1 in genus 2)**.**

$$\begin{aligned}O_2(K_X,\beta+\beta_0,1) = &(-1)^e(e-1)R_{2,(e-1,1)}(X(\log E),\beta)\\ &+\frac{1}{24}(-1)^{e+1}(e-1)R_{1,(e-1,1)}(X(\log E),\beta)\\ &+\frac{(-1)^{e+1}(10e^3-30e^2+57e-37}{5760}R_{0,(e-1,1)}(X(\log E),\beta)\\ &-\frac{1}{24}\delta_1(\pi^*\beta-C)+\Delta^{ol}(2,\beta)\end{aligned}$$

6.5. Computational validity of open-log conjecture

For $X=\mathbb{P}^2$, we give computational validity of Conjecture 6 in low genus and various degrees. Let $H\in H_2(\mathbb{P}^2,\mathbb{Z})$ be the hyperplane class. For an effective curve class $\beta\in H_2^+(\mathbb{P}^2,\mathbb{Z})$, we write $\beta=dH$ for $d>0$. The computational strategy is the following: we compute higher genus two-pointed log invariants from quantized scattering diagram of $(\mathbb{P}^2,E)$ (see Section 3.2 for more details) and compare them with open Gromov-Witten invariants of a single outer, AV-brane in low degrees and genus, which were computed by the work of Graber and Zaslow in [**GZ**].

To compute the log invariants, recall from Definition 35 that for an effective curve class $\beta\in H_2^+(X,\mathbb{Z})$,

$$\sum_{h\in\mathfrak{T}_{p,q}(X,\beta,P)} m_h(\mathbf{q}) = \sum_{g\geq 0} R^{trop}_{g,(e-1,1)}(X,\beta)\hbar^{2g}$$

We compute $R^{trop}_{g,(e-1,1)}$ in the scattering diagram consistent to order t^{12} (Figure 3.5) using the Sage code of Tim Gräfnitz.

The tropical/holomorphic correspondence of [**Gra**] (Equation 6.1) tells us that,

$$R_{g,(e-1,1)}(X,\beta) = (e-1)R^{trop}_{g,(e-1,1)}(X,\beta)$$

Using the above relations and known numbers for open Gromov-Witten invariants, we computationally verify Conjecture 6 in low degrees and genus.

d	$\sum_{g\geq 0} R^{trop}_{g,(3d-1,1)}(\mathbb{P}^2, dH)\hbar^{2g}$
1	$\mathbf{q}^{\frac{1}{2}} + \mathbf{q}^{\frac{-1}{2}}$
2	$\mathbf{q}^{-2} + \mathbf{q}^{-1} + 1 + \mathbf{q} + \mathbf{q}^2$
3	$\mathbf{q}^{9/2} + 3\mathbf{q}^{7/2} + 4\mathbf{q}^{5/2} + 4\mathbf{q}^{3/2} + 4\mathbf{q}^{1/2} + (\mathbf{q} \to \mathbf{q}^{-1})$
4	$\mathbf{q}^8 + 3\mathbf{q}^7 + 9\mathbf{q}^6 + 17\mathbf{q}^5 + 23\mathbf{q}^4 + 25\mathbf{q}^3 + 26\mathbf{q}^2 + 26\mathbf{q} + 26 + (\mathbf{q} \to \mathbf{q}^{-1})$

Table 6.1. The count of **q**-refined tropical curves in $\mathbb{P}^2$ up to degree $d = 4$.

6.5.1. Genus 1

In genus 1, recall that from Conjecture 7, we have that,

$$O_1(K_{\mathbb{P}^2}, dH + \beta_0, 1) = (-1)^{3d}(3d-1)R_{1,(3d-1,1)}(\mathbb{P}^2, dH) + \frac{(-1)^{3d}}{24}(3d-1)^3 R_{0,(3d-1,1)}(\mathbb{P}^2, dH)$$
$$+ \delta_1(\pi^* dH - C)$$

We have the following table of genus 1 invariants, We refer to A.1.1 in Appendix A

d	$O_1(K_{\mathbb{P}^2}, dH + \beta_0, 1)$	$R_{1,(3d-1,1)}(\mathbb{P}^2, dH)$	$R_{0,(3d-1,1)}(\mathbb{P}^2, dH)$	$\delta_1(\pi^* dH - C)$
1	$\frac{-1}{12}$	$\frac{-1}{8}$	1	0
2	$\frac{5}{24}$	-1	1	0
3	$\frac{23}{3}$	$\frac{-92}{8}$	4	1
4	$\frac{-3313}{12}$	$\frac{-1683}{11}$	$\frac{286}{11}$	-35

Table 6.2. Genus 1 open and log invariants for $\mathbb{P}^2$.

for computing $\delta_1(\pi^* dH - C)$ for $d \leq 4$.

In degree $d = 1$, the open-log conjecture is,

$$O_1(K_{\mathbb{P}^2}, 1, 1) = -2R_{1,(2,1)}(\mathbb{P}^2, H) - \frac{1}{3}R_{0,(2,1)}(\mathbb{P}^2, H)$$

and indeed, $\frac{-1}{12} = -2 \cdot \frac{-1}{8} - \frac{1}{3} \cdot 1$.

In degree $d = 2$, the open-log conjecture is,

$$O_1(K_{\mathbb{P}^2}, 2, 1) = 5R_{1,(5,1)}(\mathbb{P}^2, 2H) + \frac{125}{24}R_{0,(5,1)}(\mathbb{P}^2, 2H)$$

and indeed, $\frac{5}{24} = 5 \cdot -1 + \frac{125}{24} \cdot 1$.

In degree $d = 3$, the open-log conjecture is,

$$O_1(K_{\mathbb{P}^2}, 3, 1) = -8R_{1,(8,1)}(\mathbb{P}^2, 3H) - \frac{512}{24}R_{0,(8,1)}(\mathbb{P}^2, 3H) + 1$$

and indeed, $\frac{23}{3} = -8 \cdot \frac{-92}{8} - \frac{512}{24} \cdot 4 + 1$

In degree $d = 4$, the open-log conjecture is,

$$O_1(K_{\mathbb{P}^2}, 4, 1) = 11R_{1,(11,1)}(\mathbb{P}^2, 4H) + \frac{1331}{24}R_{0,(11,1)}(\mathbb{P}^2, 4H)$$
$$- 3R_{0,(3)}(\mathbb{F}_1, \pi^* H) - 2R_{0,(1)}(\mathbb{F}_1, C)R_{0,(2)}(\mathbb{F}_1, F)$$

and indeed, $\frac{-3313}{12} = 11 \cdot \frac{-1683}{11} + \frac{1331}{24} \cdot \frac{286}{11} - 3 \cdot 9 - 2 \cdot 1 \cdot 4$.

6.5.2. Genus 2

In genus 2, we have the following table of invariants,

d	$O_2(K_{\mathbb{P}^2}, dH+\beta_0,1)$	$R_{2,(3d-1,1)}(\mathbb{P}^2,dH)$	$\Delta(2,dH)$
1	$\frac{-7}{2880}$	$\frac{-1}{8}$	0
2	$\frac{7}{1152}$	-1	0
3	$\frac{-149}{360}$	$\frac{-92}{8}$	1

Table 6.3. Genus 2 open and log invariants.

The genus 2 open-log conjecture is Conjecture 8, and we have,

$$\begin{aligned} O_2(K_X, \beta+\beta_0, 1) = (e-1)^2 R_{2,(e-1,1)}(X,\beta) \\ + \frac{1}{24}(-1)^{e+1}(e-1)R_{1,(e-1,1)}(X,\beta) \\ + \frac{(-1)^{e+1}(e-1)(10e^2-20e+37)}{5760} R_{0,(e-1,1)}(X,\beta) \\ - \frac{1}{24}\delta_1(\pi^*\beta - C) - (e-1)\Delta^{ol}(2,\beta) \end{aligned}$$

The following equations were obtained from the form of genus 2 log-local given in Theorem 27.

In degree $d=1$, the open-log conjecture is,

$$O_2(K_{\mathbb{P}^2},1,1) = -2R_{2,(2,1)}(\mathbb{P}^2,H) - \frac{1}{3}R_{1,(2,1)}(\mathbb{P}^2,H) - \frac{112}{2880}R_{0,(2,1)}(\mathbb{P}^2,H)$$

and indeed, $\frac{-7}{2880} = -2\cdot\frac{1}{384} - \frac{1}{3}\cdot\frac{-1}{8} - \frac{112}{2880}\cdot 1$.

In degree $d=2$, the open-log conjecture is,

$$O_2(K_{\mathbb{P}^2},2,1) = 5R_{2,(5,1)}(\mathbb{P}^2,2H) + \frac{125}{24}R_{1,(5,1)}(\mathbb{P}^2,2H) + \frac{4375}{1152}R_{0,(2,1)}(\mathbb{P}^2,2H)$$

and indeed, $\frac{7}{1152} = 5\cdot\frac{17}{60} + \frac{125}{24}\cdot(-1) + \frac{4375}{1152}\cdot 1$.

In degree $d = 3$, the open-log conjecture is,

$$O_2(K_{\mathbb{P}^2}, 3, 1) = -8R_{2,(8,1)}(\mathbb{P}^2, 3H) - \frac{64}{3}R_{1,(8,1)}(\mathbb{P}^2, 3H) - \frac{114688}{2880}R_{0,(8,1)}(\mathbb{P}^2, 3H) - \frac{1}{24}$$

and indeed, $\frac{-149}{360} = -8 \cdot \frac{1037}{96} - \frac{64}{3} \cdot \frac{-92}{8} - \frac{114688}{2880} \cdot 4 - \frac{1}{24}$.

6.6. Quantum Theta Functions and Open Mirror Symmetry

6.6.1. A higher genus discrepancy between theta functions and mirror maps

We describe an application of Conjecture 6 and Theorem 22 to the relationship between quantum theta functions and open mirror symmetry. Theta functions are of great interest and first appeared in the theory of abelian varieties. In Gross-Siebert mirror symmetry, theta functions were constructed using broken lines in the mirror toric degeneration. Recall the definition of theta functions $\theta_q(\mathbf{q})$ for $\mathbb{P}^2$ given in Section 3.3.6.

When $q = \mathbf{q} = 1$, the classical theta function θ_1 is equivalent to the Landau-Ginzburg superpotential W after wall crossing to an unbounded chamber of the scattering diagram. After a change of variables $Q = -t^3y^3$, it is proven that,

$$\theta_1 = M(Q)$$

where $M(Q) = 1 - 2Q + 5Q^2 - 32Q^3 + \ldots$ is the open mirror map of an outer Aganagic-Vafa brane [**GRZ**], whose coefficients are open invariants of a moment fiber in local $\mathbb{P}^2$.

In higher genus, we consider a replacement of $M(Q)$ given by the generating series (Equation 9.10 of [**AKMV**]),

$$\hat{f}_{\square\emptyset\emptyset} = 1 - 2Q + 5Q^2 - (32 + 9z)Q^3 + (286 + 288z + 108z^2 + 14z^3)Q^4 + \dots \tag{6.20}$$

where $z = (\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}})^2$. The coefficient of $z^g Q^d$ is the *negative* of the genus g, degree d, LMOV invariant [1] of an outer AV-brane in local $\mathbb{P}^2$. The subscripts of $\hat{f}_{\square\emptyset\emptyset}$ are Young Tableaus for $U(N)$-representations attached to branes. Note that in the limit as $z \to 0$ or $\mathbf{q} \to 1$, we recover the open mirror map $M(Q)$.

It is natural to consider if there is an extension of [**GRZ**] to higher genus. To obtain higher genus information, we quantize the scattering diagram with wall functions as quantum cluster transformations (Section 3.1) and consider a quantized superpotential $W(\mathbf{q})$ by counting $\mathbf{q}$-refined broken lines. We wall cross $W(\mathbf{q})$ (see Section 3.2.9) to an unbounded chamber in the quantized scattering diagram to obtain $\theta_1(\mathbf{q})$, and ask if,

$$\theta_1(\mathbf{q}) = \hat{f}_{\square,\cdot,\cdot} \tag{6.21}$$

However, Equation 6.21 is seen to not hold by comparing invariants in low degrees, and Conjecture 6 describes the precise discrepancy between the open vs. log/tropical invariants in general degree and genus.

[1] In winding 1 for a single brane, the LMOV invariant is equal to the open BPS invariant n_g^{open} defined in Section 4. In general, LMOV invariants are related to open BPS invariants by a linear transformation defined by characters of symmetric group [**AKMV**].

6.6.2. Quantum theta functions from periods

We outline an alternative method to obtain quantum theta functions given by the scattering diagram of (X, E) by computing quantum periods in the context of mirror symmetry for the toric Calabi-Yau 3-fold K_X.

6.6.2.1. The 4D potential for local $\mathbb{P}^2$. Aganagic and Vafa showed that the superpotential $\mathscr{W}$ of all winding, disc invariants of AV-branes in toric Calabi-Yau-threefolds is computed by an Abel-Jacobi type map on the mirror curve [**AV**],

$$\mathscr{W} = \int y dx$$

Considering 4D effective theories, Lerche and Mayr derive Picard-Fuchs differential equations in both open and closed variables [**LM**], and compute the resulting superpotential $\mathcal{W}$ (which is different from the Landau-Ginzburg potential W, or the $\mathscr{W}$ computed by Aganagic-Vafa). For local $\mathbb{P}^2$, $\mathscr{W}$ is expressed as,

$$\mathcal{W} = \sum_{n>m\geq 0, n\geq 3m} (-1)^m \frac{(n-m-1)!}{n(n-3m)!(m!)^2} x^n z^m$$

where x and z are the open and closed complex parameters, respectively. The variable z is related to the closed symplectic parameter Q by,

$$Q = ze^{F(z)} = z - 6z^2 + 63z^2 - 866z^4 + 13899z^5 - 246366z^6 + \dots$$

The variable x is related in the open mirror map to the open and closed symplectic parameters U and Q, respectively, by the series,

$$x = UM(Q) = U - 2UQ + 5UQ^2 - 32UQ^3 + 286UQ^4 - \dots \tag{6.22}$$

Following Aganagic-Vafa, $\mathscr{W}$ should be the generating function of all winding disc invariants. Substituting the mirror maps into $\mathscr{W}$ for x and z, the coefficient of U^wQ^d is the winding w, degree d disc invariant of an Aganagic-Vafa brane. Specializing to winding $w = 1$, we recover Equation 6.22. In other words, the open mirror map is also the generating function of winding 1, disc invariants.

The function $F(z)$ is the holomorphic part of the logarithmic solution of the closed Picard-Fuchs equations for local $\mathbb{P}^2$, and $G(z)$ is the holomorphic part of the logarithmic solution of the equations derived in [**LM**]. They satisfy the relation $G(z) = \frac{-1}{3}F(z)$.

Remark 20. *The mirror map $M(Q)$ is also the generating function of open invariants of a moment fiber. We use the fact that counts of Aganagic-Vafa branes in framing 0 agree with counts of the moment fiber. Consider the generating function $F(Q) = \sum_{\beta\in NE(X)} n_{\beta+\beta_0}Q^\beta$ of disc invariants with boundary on a moment fiber in disc class $\beta+\beta_0$. Inverting the closed mirror map $z = z(Q)$ and writing $Q = Q(z)$, the series $F(Q(z))$ is a solution to the open Picard-Fuchs equations of* [**LM**]*. This implies that $F(Q(z))$ is the open mirror map $M(Q)$. In* [**FL**]*, the open invariants of a moment fiber are shown to equal the open invariants of the AV-brane.*

6.6.2.2. Quantum periods of mirror curves. The mirror of a toric Calabi-Yau threefold is described by the affine equation,

$$uv = p(x,y) \subset (\mathbb{C}^*)^4$$

where the algebraic variety $p(x,y) = 0 \subset (\mathbb{C}^*)^2$ is called the *mirror curve.* Let $X = e^x$ and $Y = e^y$. By the *quantum mirror curve* $\hat{p}(X,Y)$, we mean the ideal generated by $p(X,Y)$ in the quantum torus $[X,Y] = \hbar$. In the Weyl representation, X and Y act on functions on the quantum torus by multiplication and translation, i.e. $(X \cdot f)(X) = Xf(X)$ and $(Y \cdot f)(X) = f(qX)$. A wavefunction Ψ is a state in the Hilbert space obtained from quantization of the moduli space of complex structures. A *wavefunction* $\Psi(X)$ is by definition a function on the quantum torus that satisfies $\hat{p}(X,Y)\Psi = 0$. For example, the mirror family of genus 1, hyperelliptic curves for local $\mathbb{P}^2$ is given by the equation,

$$p(x,y) = 1 - x - y - \frac{z}{xy}$$

where z is a complex modulus of the curve. The quantum mirror curve for local $\mathbb{P}^2$ is,

$$\hat{p}(X,Y) = 1 - X - Y - \mathbf{q}^{\frac{1}{2}} z X^{-1} Y^{-1}$$

By WKB approximation, we have $\Psi \sim \exp(\mathscr{W}/\hbar)$, which to leading order is the 4D superpotential $\mathscr{W}$. The wavefunction contains higher genus information from the A-model. As a result of computational validity of Conjecture 6, wavefunctions defined by log/tropical invariants do not agree with those defined by open invariants.

From [**ACDKV**], the quantum A-periods $a(z,\mathbf{q})$ of the mirror curve are defined as the residue,

$$a(z,\mathbf{q}) := res\left(\frac{1}{X}\log\frac{\Psi(\mathbf{q}X)}{\Psi(X)}\right) \tag{6.23}$$

These are computed by using the condition $\hat{p}(X,Y)\Psi(X) = 0$ and solving a difference equation obtained from the mirror curve. Using $a(z,\mathbf{q})$ to define quantum periods and hence a quantized mirror map, we observed that the resulting coefficients match the two-legged tropical invariants $R^{trop}_{(e-1,1)}(\mathbb{P}^2,\beta,\mathbf{q})$.

6.6.3. Periods from Mirror Symmetry for Fano manifolds

We briefly outline mirror symmetry for Fano manifolds X as described in [**CCGGK**], which relates certain descendant Gromov-Witten invariants of X to periods of the Landau-Ginzburg superpotential on the mirror space. The potential is given by a Laurent polynomial $f : (\mathbb{C}^*)^n \to \mathbb{C}$. The *classical A-period* $\pi_f(t)$ of f is defined as,

$$\pi_f(t) := \frac{1}{(2\pi i)^n}\int_{|x_i|=1}\frac{1}{1-tf}\frac{dx_1}{x_1}\wedge\ldots\wedge\frac{dx_n}{x_n}$$

By the residue theorem, we can write,

$$\pi_f(t) = \sum_{m=0}^{\infty} c_m t^m$$

with c_m defined as the constant coefficient of f^m. The periods π_f form a basis of the solutions for Picard-Fuchs differential equations.

Example 23. *Let $X = \mathbb{P}^2$. The Landau-Ginzburg potential is given by $f(x,y) = x + y + x^{-1}y^{-1}$. The periods of f are solutions to the differential equation,*

$$[D^2 - 27t^3(D+1)(D+2)]\pi_f = 0$$

where $D = t\frac{d}{dt}$*. This implies that the coefficients of* $\pi_f(t)$ *satisfy the recursion relation,*

$$m^2 c_{3m} - 3(3m-1)(3m-2)c_{3m-3} = 0$$

Therefore, $\pi_f(t) = \sum_{m\geq 0} \frac{(3m)!}{(m!)^3} t^{3m}$.

To define the *quantum periods* of a Fano manifold, we consider its descendant Gromov-Witten invariants. Let $\overline{\mathcal{M}}_{0,1}(X,\beta)_m$ be the moduli space of genus 0 stable maps $f : (C,x) \to X$ with 1 marked point to X such that $c_1(\beta) = m$. The quantum period $G_X(t)$ is defined as,

$$G_X(t) := \sum_{m\geq 0} p_m t^m \tag{6.24}$$

with $p_0 := 1, p_1 := 0$, and $p_m := \int_{[\overline{\mathcal{M}}_{0,1}(X,\beta)_m]^{vir}} \psi^{m-2} ev^*[pt]$, where ψ is the first Chern class of the cotangent line at the marked point x, and $[pt] \in H^4(X)$ is the Poincaré dual of a point. Mirror symmetry for Fano manifolds can be defined as X is mirror to f if we have,

$$\pi_f(t) = G_X(t)$$

up to renormalization constants (see Definition 4.9, [**CCGGK**]).

6.6.4. [GRZZ]

In work in progress [**GRZZ**], we give a proof that the quantum A-period in Equation 6.23 agrees with the quantum Gromov-Witten period in Equation 6.24, by considering higher valency tropical curves. We also consider open-log correspondences in higher winding.

CHAPTER 7

Open-closed BPS conjecture for toric Calabi-Yau threefolds, with results for local $\mathbb{P}^2$

In this chapter, we conjecture an equality of BPS invariants for open and closed Gromov-Witten invariants of toric Calabi-Yau threefolds. For local $\mathbb{P}^2$, we use the topological vertex and its refined version to prove the conjecture in low degrees and all genus. For local $\mathbb{P}^2$, the calculation will demonstrate that the genus g, one-holed, open BPS invariants with boundary on an Aganagic-Vafa brane are equal to certain genus g, closed BPS invariants of local $\mathbb{F}_1$.

7.1. Introduction

Recall that X is a toric Fano surface with a smooth anticanonical divisor E, and $\pi : \hat{X} \to X$ is a toric blow up with exceptional curve C. Let K_X and $K_{\hat{X}}$ be the local Calabi-Yau geometries given by the canonical bundles of X and $\hat{X}$.

Let $\beta \in H_2^+(X,\mathbb{Z})$ be an effective curve class. Let $N_{g,0}(K_{\hat{X}}, \pi^*\beta - C)$ be the genus g Gromov-Witten invariant of $K_{\hat{X}}$ with no insertions in class $\pi^*\beta - C$. Since $\pi^*\beta - C$ is a primitive curve class, the Gopakumar-Vafa formula takes the form,

$$\sum_{g\geq 0} N_g(K_{\hat{X}}, \pi^*\beta - C)\hbar^{2g-2} = \sum_{g\geq 0} n_g(K_{\hat{X}}, \pi^*\beta - C)\left(2\sin\frac{\hbar}{2}\right)^{2g-2}$$

$n_g(K_{\hat{X}}, \pi^*\beta - C)$ is the corresponding Gopakumar-Vafa invariant.

Let $O_g(K_X, \beta + \beta_0, 1)$ be the genus g, winding 1, framing 0 open Gromov-Witten invariant of a single outer AV-brane in K_X in curve class $\beta + \beta_0$. A multiple cover formula for $O_g(K_X, \beta + \beta_0, 1)$ is given by,

$$\sum_{g \geq 0} O_g(K_X, \beta + \beta_0, 1)\hbar^{2g-1} = \sum_{g \geq 0} (-1)^{g+1} n_g^{open}(K_X, \beta + \beta_0, 1) \left(2 \sin \frac{\hbar}{2}\right)^{2g-1}$$

$n_g^{open}(K_X, \beta + \beta_0, 1)$ is the corresponding open BPS invariant. We refer to Chapter 4 for the full definition of the invariants and multiple cover formulas.

In genus 0, we have the following relations from the multiple cover formulas,

$$N_0(K_{\hat{X}}, \pi^* \beta - C) = n_0(K_{\hat{X}}, \pi^* \beta - C), \quad O_0(K_X, \beta + \beta_0, 1) = -n_0^{open}(K_X, \beta + \beta_0, 1)$$

Notice that Theorem 1.1 of [**LLW**] is equivalent to,

$$n_0(K_{\hat{X}}, \pi^* \beta - C) = -n_0^{open}(K_X, \beta + \beta_0, 1) \tag{7.1}$$

In higher genus, closed BPS invariants $n_g(K_{\hat{X}}, \pi^* \beta - C)$ were first computed by localization in [**KZ**] for $X = \mathbb{P}^2$. They can also be computed by the topological vertex [**AKMV**], Gromov-Witten/Donaldson-Thomas correspondence [**MNOP**], or Eynard-Orantin recursion and mirror symmetry [**FLZ**], [**FRZZ**]. In [**HKR**], the authors use integrability of the holomorphic anomaly equation to compute closed BPS invariants of many local Calabi-Yau geometries.

A rigorous definition of higher genus open string invariants is still elusive. Graber and Zaslow computed higher genus open string invariants by assuming an open string virtual localizaiton theorem, and matched the physical predictions of Aganagic-Vafa [**GZ**]. Open BPS invariants were computed by large-N duality in [**AKMV**].

By directly comparing invariants computed by [**GZ**] and [**HKR**], we conjecture,

Conjecture 9 (Conjecture 3)**.** *Let X be a toric del Pezzo surface, and $\pi : \hat{X} \to X$ a toric blow up with exceptional curve C. Then we have the following equality,*

$$n_g(K_{\hat{X}}, \pi^*\beta - C) = (-1)^{g+1} n_g^{open}(K_X, \beta + \beta_0, 1)$$

where $n_g(K_{\hat{X}}, \pi^*\beta - C)$ is the genus g, closed Gopakumar-Vafa invariant of the canonical bundle $K_{\hat{X}}$ in curve class $\pi^*\beta - C$, and $n_g^{open}(K_X, \beta + \beta_0, 1)$ be the genus g, 1-holed, winding 1, open BPS invariant of K_X with boundary on a single, outer Aganagic-Vafa brane in framing 0 in disc class $\beta + \beta_0 \in H_2(K_X, L)$.

Notice that Conjecture 9 seeks to extend the $g = 0$, open-closed result of [**LLW**] (Equation 7.1).

In Section 7.3, assuming a mathematical validity of higher genus open string invariants, we use the topological vertex to show the following,

Theorem 24 (Theorem 6)**.** *Conjecture 9 is true for $X = \mathbb{P}^2$ in curve classes $\beta = dH$ for $d = 1, 2, 3, 4$ and in all genus.*

7.2. Preliminaries of the Topological Vertex

The topological vertex of [**AKMV**] computes all genus topological string amplitudes for toric Calabi-Yau threefolds by leveraging large N duality between topological string theory and Chern-Simons theory.

The work of [**LLLZ**] puts the topological vertex on a rigorous mathematical footing. The 1-leg topological vertex was proven to be correct in [**LLZ1**], [**OP2**]. The 2-leg vertex was proven in [**LLZ2**]. In [**MOOP**], they prove the full 3-leg vertex is correct. Thus, the ensuing computations provide a valid proof for Conjecture 9 in the degrees calculated. In degree 1, we prove the conjecture by directly computing partitions functions with the topological vertex. In degree 2, 3, 4, we use computations from the refined topological vertex [**IKV**].

7.2.1. Definitions

We describe some formalism of the topological vertex. We refer to [**Kon**], [**LLLZ**], [**AMV**], [**AKMV**] for more details.

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers containing only finitely many nonzero terms. Recall that representations R of $U(N)$ are labelled by Young tableaus, which is simply a partition λ with λ_i boxes in the i-th row. We write λ^R for the Young tableau associated to a representation R. We will talk about a representation R and its Young tableau λ^R interchangeably. The transpose Young diagram λ^T has λ_i boxes in the i-th column, and we write λ_i^T as the number of boxes in the i-th row of λ^T. Define the *length* $\ell(\lambda)$ to be the number of nonzero λ_i. Let $|\lambda| = \sum_i \lambda_i$ be the *weight* of λ. Define the quantity,

$$\kappa_\lambda := \sum_i \lambda_i(\lambda_i - 2i + 1)$$

We write $\varnothing$ for the empty Young Tableau.

Next, we define Schur functions in terms of the elementary symmetric functions. For a sequence of variables $x = (x_1, x_2, \ldots)$, recall that the elementary symmetric functions $e_i(x), i \geq 0$ are defined by the product,

$$\sum_{i=0}^{\infty} e_i(x) z^i := \prod_{i=0}^{\infty} (1 + x_i z)$$

Given a partition λ, the Schur function $s_\lambda(x)$ is given by,

$$s_\lambda(x) := \det(e_{\lambda_i^T - i + j}(x))_{1 \leq i,j \leq |\lambda|}$$

This is known as the *Jacobi-Trudi formula* for Schur functions.

Given two partitions $\mu \subset \lambda$, i.e $\mu_i \leq \lambda_i$ for all i, we similarly define the *skew Schur function* $s_{\lambda/\mu}(x)$ as,

$$s_{\lambda/\mu}(x) := \det(e_{\lambda_i^T - \mu_j^T - i + j}(x))_{1 \leq i,j \leq |\lambda|}$$

When $\mu = \varnothing$, we recover the usual Schur function $s_\lambda(x)$.

We use q as a formal variable. Let λ be a partition. We define two infinite sequences,

$$q^\rho := (q^{-i+\frac{1}{2}})_{i \geq 1}, \quad q^{\lambda+\rho} := (q^{\lambda_i - i + \frac{1}{2}})_{i \geq 1}$$

Given three partitions λ, μ, ν, we define *the topological vertex* $C(\lambda, \mu, \nu)$ as,

$$C(\lambda, \mu, \nu) = q^{(\kappa_\lambda + \kappa_\nu)/2} s_{\nu^T}(q^\rho) \sum_\eta s_{\lambda^T/\eta}(q^{\nu+\rho}) s_{\mu/\eta}(q^{\nu^T+\rho})$$

The arguments of the topological vertex are $\mathbb{Z}/3$-symmetric, i.e.

$$C(\lambda,\mu,\nu) = C(\nu,\lambda,\mu) = C(\mu,\nu,\lambda)$$

We recall from Section 6.6, Equation 6.20 the generating series (and setting $y = Q$),

$$\hat{f}_{\Box\varnothing\varnothing} = 1 - 2y + 5y^2 - (32 + 9z)y^3 + (286 + 288z + 108z^2 + 14z^3)y^4 - \ldots \tag{7.2}$$

with $z = (\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}})^2$ and $y = e^{-t}$. The coefficient of $z^g y^d$ is the *negative* of the genus g, degree d, 1-holed, open BPS invariant of local $\mathbb{P}^2$ in winding 1. We refer to [**AKMV**], Section 7.3 for the procedure to obtain $\hat{f}$ from the topological vertex partition function.

7.3. Proof of Theorem 6

Let $X = \mathbb{P}^2$ and $\hat{X} = \mathbb{F}_1$. We use the topological vertex [**AKMV**] and its refined version [**IKV**] to verify Conjecture 9 for the curve class $\beta = dH$ for $d = 1,2,3,4$ and for all $g \geq 0$.

7.3.1. Degree 1

We calculate the partition functions of local $\mathbb{P}^2$ and local $\mathbb{F}_1$ using the topological vertex.

7.3.1.1. Closed free energy of Local $\mathbb{F}_1$. Recall that the Kähler cone of $\mathbb{F}_1$ is given by B and F, where $B^2 = -1, B \cdot F = 1$, and $F^2 = 0$. We will write $H := B + F$ which satisfies $H^2 = 1$. We denote t_1 and t_2 to be the Kähler parameters corresponding to the area of B and F, respectively. In degree 1, the curve class is given by $F = H - B$. The anticanonical class of $\mathbb{F}_1$ is given by $2F + B + H$.

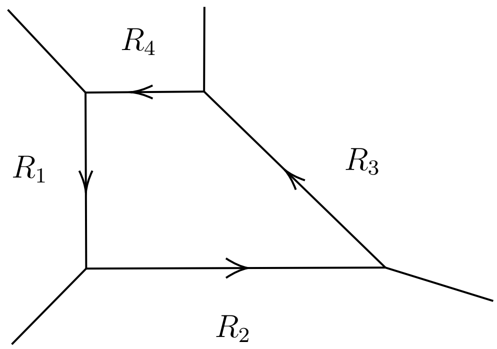


Figure 7.1. The toric diagram of local $\mathbb{F}_1$ with representations R_1 and R_3 attached to the two fibers F, R_2 attached to H, R_4 attached to B, and no representations attached to external legs.

We label each bounded edge with a representation R_i, and each unbounded edge with the trivial representation, as we do not consider any D-branes. We attach two representations R_1 and R_3 to the edges corresponding to the class F, the representation R_2 to the edge for H, and representation R_4 to the edge for B. We traverse the toric graph in a counterclockwise direction. We sum over Young tableau λ_i of R_i. The partition function $Z_{K_{\mathbb{F}_1}}$ for local $\mathbb{F}_1$ is given by,

$$\begin{aligned} Z_{K_{\mathbb{F}_1}} &= \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} (-1)^{\sum_i |\lambda_i|} e^{-(|\lambda_1|+|\lambda_3|)t_2} e^{-|\lambda_4| t_1} e^{-|\lambda_2|(t_1+t_2)} q^{\sum_i \kappa_{\lambda_i}} C_{\varnothing\lambda_1\lambda_4^T} C_{\varnothing\lambda_2\lambda_1^T} C_{\varnothing\lambda_3\lambda_2^T} C_{\varnothing\lambda_4\lambda_3^T} \\ &= \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} (-1)^{\sum_i |\lambda_i|} e^{-(|\lambda_4|+|\lambda_2|)t_1} e^{-(|\lambda_1|+|\lambda_3|+|\lambda_2|)t_2} q^{\sum_i \kappa_{\lambda_i}} C_{\varnothing\lambda_1\lambda_4^T} C_{\varnothing\lambda_2\lambda_1^T} C_{\varnothing\lambda_3\lambda_2^T} C_{\varnothing\lambda_4\lambda_3^T} \end{aligned}$$

We take the logarithm of $Z_{\mathbb{F}_1}$ to obtain the free energy $F_{\mathbb{F}_1}$ encoding the BPS degeneracies. Hence, we write,

$$F_{K_{\mathbb{F}_1}} = \log Z_{K_{\mathbb{F}_1}} = \log\left(1 + \sum_{\substack{\ell_1,\ell_2=0,\\ \ell_1+\ell_2\neq 0}}^{\infty} a_{\ell_1,\ell_2}(q) e^{-\ell_1 t_1 - \ell_2 t_2}\right)$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\left[\sum_{\substack{\ell_1,\ell_2=0,\\ \ell_1+\ell_2\neq 0}}^{\infty} a_{\ell_1,\ell_2}(q) e^{-\ell_1 t_1 - \ell_2 t_2}\right]^k}{k}$$

$$= \sum_{\substack{\ell_1,\ell_2=0,\\ \ell_1+\ell_2\neq 0}}^{\infty} a^{(c)}_{\ell_1,\ell_2}(q) e^{-\ell_1 t_1 - \ell_2 t_2}$$

where $a_{\ell_1,\ell_2}(q)$ and $a^{(c)}_{\ell_1,\ell_2}(q)$ are functions defined by taking the logarithm. The coefficients $a^{(c)}_{\ell_1,\ell_2}(q)$ encode the Gromov-Witten invariants.

In degree 1 or $(\ell_1, \ell_2) = (0, 1)$, we find the invariants given by $a^{(c)}_{(0,1)}(q)$. This implies that $\lambda_2 = \lambda_4 = \varnothing$, and two contributions corresponding to $\lambda_1 = \square$ and $\lambda_2 = \lambda_3 = \lambda_4 = \varnothing$, or $\lambda_3 = \square$ and $\lambda_1 = \lambda_2 = \lambda_4 = \varnothing$. Thus,

$$a^{(c)}_{0,1}(q) = -2C^2_{\square\varnothing\varnothing} = \frac{-2}{(q^{\frac{1}{2}} - q^{\frac{-1}{2}})^2}$$

(see [**AKMV**], pg. 53 for a list of formulas for the vertex functions). By the closed Gopakumar-Vafa formula (Equation 4.4), this implies that,

$$n_0(K_{\mathbb{F}_1}, F) = -2, \quad n_g(K_{\mathbb{F}_1}, F) = 0 \text{ for } g > 0 \tag{7.3}$$

7.3.1.2. Open free energy of local $\mathbb{P}^2$. We consider a single outer Aganagic-Vafa brane in local $\mathbb{P}^2$, and we attach the representation $Q = \square$ to it. The toric diagram of local $\mathbb{P}^2$ is,

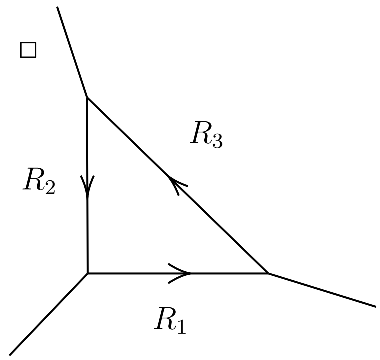


Figure 7.2. The toric diagram of local $\mathbb{P}^2$, with representations R_i attached to the edges corresponding to the hyperplane class $H \in H_2(\mathbb{P}^2, \mathbb{Z})$, and a representation □ attached a single external D-brane.

The open partition function is given by,

$$Z_{K_{\mathbb{P}^2}}(V) = \sum_{\lambda_1,\lambda_2,\lambda_3} (-1)^{\sum_i |\lambda_i|} e^{-\sum_i |\lambda_i| t} q^{\sum_i \kappa_{\lambda_i}} C_{\square\lambda_2\lambda_3^T} C_{\varnothing\lambda_1\lambda_2^T} C_{\varnothing\lambda_3\lambda_1^T} Tr_{\square} V$$

(see [**AKMV**], Equation 9.9). Here the input V is a holonomy corresponding to a $U(N)$ matrix, and $Tr_\square V$ means the trace of V in the $U(N)$-representation □. The value of $Tr_\square V$ is the open symplectic parameter of the brane.

We find the partition function in degree 1, i.e. when $\sum_i |\lambda_i| = 1$. There are 3 possibilities, corresponding to,

(1) $\lambda_1 = \square, \lambda_2 = \lambda_3 = \varnothing$

(2) $\lambda_2 = \square, \lambda_1 = \lambda_3 = \varnothing$

(3) $\lambda_3 = \square, \lambda_1 = \lambda_2 = \varnothing$

In case 1), the contribution to $Z_{K_{\mathbb{P}^2}}(V)$ is,

$$-e^{-t} C^3_{\square\varnothing\varnothing} = \frac{-e^{-t}}{(q^{\frac{1}{2}} - q^{\frac{-1}{2}})^3}$$

In case 2), the contribution is,

$$-e^{-t}C_{\square\square\varnothing}C_{\square\varnothing\varnothing} = -e^{-t}\frac{q^2-q+1}{(q-1)^2(q^{\frac{1}{2}}-q^{\frac{-1}{2}})}$$

(see [**AKMV**], pg. 53 for a list of formulas for the vertex functions). The total contribution is,

$$A := -e^{-t}\left[\frac{1}{(q^{\frac{1}{2}}-q^{\frac{-1}{2}})^3}+\frac{2(q^2-q+1)}{(q-1)^2(q^{\frac{1}{2}}-q^{\frac{-1}{2}})}\right] = -e^{-t}\left[\frac{2}{q^{\frac{1}{2}}-q^{\frac{-1}{2}}}+\frac{3}{(q^{\frac{1}{2}}-q^{\frac{-1}{2}})^3}\right]$$

Following [**AKMV**], Section 7.3, we consider $(q^{\frac{1}{2}}-q^{\frac{-1}{2}})A$ in the case of winding 1 and a single AV-brane.

To find the open BPS invariants, we want to obtain the open free energy "properly understood" (see the paragraph right below Equation (9.9) of [**AKMV**]). In other words, we subtract by the closed free energy $\log Z_{\mathbb{P}^2}$ of local $\mathbb{P}^2$, and find the quantity.

$$\log Z_{K_{\mathbb{P}^2}}(V) - \log Z_{\mathbb{P}^2}$$

But, the q-coefficient of $-e^{-t}$ in $\log Z_{\mathbb{P}^2}$ is precisely $\frac{3}{\left(q^{\frac{1}{2}}-q^{\frac{-1}{2}}\right)^2}$ (see [**AMV**], Equation 7.42).

Multiplying by -1 to obtain open BPS invariants (see Equation 7.2, or Equation 6.20), we have computed that,

$$n_0(K_{\mathbb{P}^2}, H+\beta_0, 1) = 2, \quad n_g(K_{\mathbb{P}^2}, H+\beta_0, 1) = 0 \text{ for } g > 0 \tag{7.4}$$

where $H \in H_2(\mathbb{P}^2, \mathbb{Z})$ is the hyperplane class, and $\beta_0 \in H_2(K_{\mathbb{P}^2}, L)$ is a holormophic disc class.

Thus, it is clear from comparing Equations 7.3 and 7.4 that Conjecture 9 holds in degree 1 for $\mathbb{P}^2$.

7.3.2. Degrees 2, 3, and 4

In degrees $2, 3$, and 4, we use formalism from the refined topological vertex [**IKV**].

7.3.2.1. Context. M-theory compactification on a toric Calabi-Yau threefold X gives a certain gauge theory that is a function of two equivariant parameters ϵ_1 and ϵ_2. In the limit $\epsilon_1 = -\epsilon_2$, the compactified theory reduces to A-model topological string theory on X which is computed by the topological vertex [**AKMV**]. The *refined topological vertex* is concerned in the setting when $\epsilon_1 \neq -\epsilon_2$. The refined vertex introduces an additional parameter t into the vertex functions $C_{R_1 R_2, R_3}(q,t)$. The refined vertex is only applicable to geometries that give rise to gauge theories, hence the refined partition function can be found for local $\mathbb{F}_1$, but not local $\mathbb{P}^2$.

Particles are represented by irreducible representations of $SO(4) \cong SU(2)_L \times SU(2)_R$, which are labelled by their right and left spin j_L and j_R respectively. For a given charge $Q \in H_2(X, \mathbb{Z})$, the D-brane moduli space can be described as the moduli space $\mathcal{M}$ of stable sheaves F on X with $c_1(F) = Q$. The BPS degeneracies of particles in charge Q are given by cohomology classes of $\mathcal{M}$. From an M2-brane wrapped on a holomorphic curve C, the refined partition function computes,

$$\sum_{j_L, j_R} N_C^{(j_L, j_R)}(j_L, j_R)$$

where $N_C^{(j_L,j_R)}$ is the number of BPS states from an M2-brane wrapped on C with left-right spin content (j_L, j_R), or the number of cohomology classes with spin (j_L, j_R) of the moduli space of the D-brane.

7.3.3. Refined vertex computations

We extract the spin content in degrees $d = 2, 3, 4$ from the refined partition function on local $\mathbb{F}_1$.

Given a tuple of left and right spins (j_L, j_R), denote $V_{j_L} \otimes V_{j_R}$ to be the corresponding $SU(2)_L \times SU(2)_R$-representation.

Definition 37. *For a representation V_s of spin $s \in \frac{1}{2}\mathbb{Z}$, we define its $\mathbf{q}$-content $\mathbf{q}_{V_s}$ to be the quantity,*

$$\mathbf{q}_{V_s} := \mathbf{q}^{-2s} + \mathbf{q}^{-2s+2} + \ldots + \mathbf{q}^{2s-2} + \mathbf{q}^{2s}$$

To compare with computations from the topological vertex, we ignore the right spin in finding the $\mathbf{q}$-content. Hence, we define the total $\mathbf{q}$-content of $V_{j_L} \otimes V_{j_R}$ to be,

$$\mathbf{q}_{V_{j_L} \otimes V_{j_R}} := (\mathbf{q}^{-2j_L} + \mathbf{q}^{-2j_L+2} + \ldots + \mathbf{q}^{2j_L-2} + \mathbf{q}^{2j_L}) \dim V_{j_R}$$

Since a vector space V_s of spin $s \in \frac{1}{2}\mathbb{Z}$ has dimension $2s + 1$, the above becomes,

$$\mathbf{q}_{V_{j_L} \otimes V_{j_R}} = (\mathbf{q}^{-2j_L} + \mathbf{q}^{-2j_L+2} + \ldots + \mathbf{q}^{2j_L-2} + \mathbf{q}^{2j_L})(2j_R + 1)$$

Given a charge $C \in H_2(\mathbb{F}_1, \mathbb{Z})$, we write $\mathbf{q}_C$ to be the total spin content of BPS degeneracies in charge C.

We show that up to a change of sign, the $\mathbf{q}$-content calculated from local $\mathbb{F}_1$ matches the coefficients of $\hat{f}_{\square\varnothing\varnothing}$. We first calculate the $\mathbf{q}$-content in the various degrees.

7.3.3.1. Degree 2. Take the charge $B+2F\in H_2(\mathbb{F}_1,\mathbb{Z})$. The spin content is $(0,2)$, with corresponding representation $V_0\otimes V_2$. Therefore, $\mathbf{q}_{V_0\otimes V_2}=5$. Notice this matches the coefficient of y^2 in $\hat{f}_{\square\varnothing\varnothing}$.

7.3.3.2. Degree 3. Take the charge $2B+3F\in H_2(\mathbb{F}_1,\mathbb{Z})$, which is a curve of genus 1. The spin content is,

$$(\frac{1}{2},4)\oplus(0,\frac{7}{2})\oplus(0,\frac{5}{2})$$

with corresponding representation,

$$V_{2B+3F}:=\left(V_{\frac{1}{2}}\otimes\mathbb{C}^9\right)\oplus\left(V_0\otimes\mathbb{C}^8\right)\oplus\left(V_0\otimes\mathbb{C}^6\right)$$

Its $\mathbf{q}$-content is given by,

$$\mathbf{q}_{V_{2B+3F}}=9(\mathbf{q}^{-1}+\mathbf{q})+14$$

Notice that $32+9z=9(\mathbf{q}^{-1}+\mathbf{q})+14$, with $z=(\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}})^2$, and hence matches the coefficient of y^3 in $\hat{f}_{\square\varnothing\varnothing}$.

7.3.3.3. Degree 4. Take the charge $3B+4F$. The spin content [1] is given by,

$$\left(\frac{3}{2},\frac{13}{2}\right)\oplus(1,6)\oplus(1,5)\oplus2\left(\frac{1}{2},\frac{11}{2}\right)\oplus(0,6)\oplus2\left(\frac{1}{2},\frac{9}{2}\right)\oplus(0,5)\oplus\left(\frac{1}{2},\frac{7}{2}\right)\oplus2(0,4)\oplus(0,3)\oplus(0,2)$$

[1]In [**IKV**], the coefficient of $(0,4)$ in the spin content is 1 instead of 2. We think that the coefficient is in fact 2, in order to recover in the limit $\mathbf{q}\to1$ the genus 0 BPS invariant of local $\mathbb{F}_1$ in class $3B+4F$, which is 286, computed in [**LLW**], [**CKYZ**], [**AKMV**].

with corresponding representation,

$$\left(V_{\frac{3}{2}}\otimes\mathbb{C}^{14}\right)\oplus\left(V_1\otimes\mathbb{C}^{13}\right)\oplus\left(V_1\otimes\mathbb{C}^{11}\right)\oplus\left(V_{\frac{1}{2}}\otimes\mathbb{C}^{12}\right)\oplus\left(V_{\frac{1}{2}}\otimes\mathbb{C}^{12}\right)\oplus\left(V_0\otimes\mathbb{C}^{13}\right)$$
$$\oplus\left(V_{\frac{1}{2}}\otimes\mathbb{C}^{10}\right)\oplus\left(V_{\frac{1}{2}}\otimes\mathbb{C}^{10}\right)\oplus\left(V_0\otimes\mathbb{C}^{11}\right)\oplus\left(V_{\frac{1}{2}}\otimes\mathbb{C}^{8}\right)\oplus 2\left(V_0\otimes\mathbb{C}^{9}\right)\oplus\left(V_0\otimes\mathbb{C}^{7}\right)\oplus\left(V_0\otimes\mathbb{C}^{5}\right)$$

Its $\mathbf{q}$-content is given by,

$$14\mathbf{q}_{V^{\frac{3}{2}}}+13\mathbf{q}_{V^1}+11\mathbf{q}_{V^1}+24\mathbf{q}_{V^{\frac{1}{2}}}+13+20\mathbf{q}_{V^{\frac{1}{2}}}+11+8\mathbf{q}_{V^{\frac{1}{2}}}+30$$

When setting $\mathbf{q}=1$, the above sum is 286. One can check with Sage that the $\mathbf{q}$-content is equal to $286+288z+108z^2+14z^3$, with $z=(\mathbf{q}^{\frac{1}{2}}-\mathbf{q}^{\frac{-1}{2}})^2$. This matches the coefficient of y^4 in $\hat{f}_{\square\varnothing\varnothing}$.

Now, define the generating series,

$$T_{\mathbb{F}_1}:=\sum_{d\geq 0}\mathbf{q}_{(d-1)B+dF}y^d$$

where we define $\mathbf{q}_{-B}:=1$ and $\mathbf{q}_F=2$, the latter number in accordance with the genus 0, Gromov-Witten invariant of local $\mathbb{F}_1$ in the class F. Express $T_{\mathbb{F}_1}$ in terms of z,

$$T_{\mathbb{F}_1}(z,y)=\sum_{d\geq 0}a_d(z)y^d$$

where each $a_d(z)$ is a linear combination of non-negative powers of z. We change sign $y\to -y$ to have agreement of $T_{\mathbb{F}_1}(z,-y)$ and $\hat{f}_{\square\varnothing\varnothing}(z,y)$ in degrees $d\leq 4$.

In order to obtain the BPS invariants from $T_{\mathbb{F}_1}(z,-y)$, we make the variable change $z \to -z$. The topological vertex tells us that the coefficient of $z^g y^d$ in $T_{\mathbb{F}_1}(-z,-y)$ is $n_g(K_{\mathbb{F}_1},(d-1)B+dF)$, or the genus g, Gopakumar-Vafa invariant in class $(d-1)B+dF$. Indeed, the first few terms are,

$$T_{\mathbb{F}_1}(-z,-y) = 1 - 2y + 5y^2 - (32-9z)y^3 + (286 - 288z + 108z^2 - 14z^3)y^4 - \dots \tag{7.5}$$

which agrees with calculations in [**HKR**], Appendix B.

Remark 21. *There is a conceptual reason for the variable change $z \to -z$. Recall that the expression $(2\sin\frac{\hbar}{2})^{2g-2} = (\mathbf{q}^{\frac{1}{2}} - \mathbf{q}^{\frac{-1}{2}})^{2g-2} = z^{g-1}$ appears in the Gopakumar-Vafa formula. The sign change $z \to -z$ multiplies the coefficients there by $(-1)^{g-1}$. After this sign change, we obtain the closed BPS invariants (see Equation 7.37 of* [**AMV**]*).*

In particular, the expression for $T_{\mathbb{F}_1}(-z,-y)$ implies that,

$$n_g(K_{\mathbb{F}_1}, B+2F) = 0, \quad \text{for } g > 0$$

$$n_g(K_{\mathbb{F}_1}, 2B+3F) = 0, \quad \text{for } g > 1$$

$$n_g(K_{\mathbb{F}_1}, 3B+4F) = 0, \quad \text{for } g > 3$$

Thus, we have,

Proof of Theorem 24. This follows from the comparison of $n_g(K_{\mathbb{F}_1},(d-1)B+dF)$ for $d \le 4$ in Equation 7.5 with the open BPS invariants in $-\hat{f}_{\square\emptyset\emptyset}$ given by the negative of Equation 7.2. $\square$

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APPENDIX A

The $g > 0$ log-local principle

We provide a short summary of the higher genus log-local principle of [**BFGW**] and specialize it to genus 1 and 2.

Let X be a del Pezzo surface and E a smooth anticanonical divisor. By the adjunction formula, E is an elliptic curve. [1], The $g > 0$ log-local principle expresses genus g, maximal tangency, relative Gromov-Witten invariants with λ_g-insertion of $X(\log E)$ in terms of local Gromov-Witten invariants of K_X and the stationary Gromov-Witten theory of E.

Theorem 25 ($g > 0$ log-local principle of [**BFGW**])**.** *For every $g \geq 0$, we have that,*

$$F_g^{K_X} = (-1)^g F_g^{X(\log E)} + \sum_{n\geq 0} \sum_{\substack{g=h+g_1+\ldots+g_n, \\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0} \\ (a_j,g_j)\neq(0,0), \sum_{j=1}^n a_j = 2h-2}} \frac{(-1)^{h-1} F^E_{h,\mathbf{a}}}{|Aut(\mathbf{a},\mathbf{g})|} \prod_{j=1}^n (-1)^{g_j-1} D^{a_j+2} F^{X/E}_{g_j}$$

We refer to [**BFGW**] for the general definitions of the generating series of local invariants $F_g^{K_X}$, the generating series of relative invariants $F_g^{X/E}$, and the generating series of elliptic curve invariants $F^E_{h,\mathbf{a}}$. The proof uses the relative virtual localization theorem of [**GV**] to compute certain invariants arising from the degeneration formula.

[1]The $g > 0$ log-local principle applies more generally to X a smooth projective variety and D a smooth nef divisor.

Remark 22. *Since E is smooth, the relative and log Gromov-Witten invariants of (X,E) agree, and henceforth we will speak of them interchangeably. We will use the definition of log/relative invariants from Section 2.6.*

In practice, we will use the following form of Theorem 25,

Corollary 4. *On the level of individual invariants, for a curve class β satisfying $\beta \cdot E > 0$, the theorem states that,*

$$N_g(K_X,\beta) = \frac{(-1)^{\beta\cdot E-1}}{\beta\cdot E} R_g(X(\log E),\beta) + \sum_{n\geq 0} \sum_{\substack{g=h+g_1+\ldots+g_n,\\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0},\\ \beta=d_E[E]+\beta_1+\ldots+\beta_n,\\ d_E\geq 0,\beta_j\cdot D>0}} \left[\frac{(-1)^{g-1+(E\cdot E)d_E}(E\cdot E)^m}{m!|Aut(\mathbf{a},g)|}\right.$$
$$\left. N_{h,(\mathbf{a},1^m)}(E,d_E)\prod_{j=1}^n\left((-1)^{\beta_j\cdot E}(\beta_j\cdot E)R_{g_j,(\beta_j\cdot E)}(X,\beta_j)\right)\right]$$

where $m := 2g-2-\sum_j a_j$, and $|Aut(\mathbf{a},g)| = |Aut(a_1,g_1)|\ldots|Aut(a_n,g_n)|$ with $|Aut(a_i,g_i)|$ being the number of partitions of a_i into g_i boxes.

In Corollary 4, the stationary invariants $N_{h,(\mathbf{a},1^m)}(E,d_E)$ of the elliptic curve for $\mathbf{a}\in\mathbb{Z}^n_{\geq 0}$ are defined as,

$$N_{h,(\mathbf{a},1^m)}(E,d_E) := \int_{[\overline{\mathcal{M}}_{h,n+m}(E,d_E)]^{vir}} \prod_{i=1}^n ev_i^*[pt]\psi_i^{a_i} \prod_{j=1}^m ev_j^*[pt]\psi_j$$

where $\psi_j \in H^2(\overline{\mathcal{M}}_{h,n+m}(E,d_E))$ is the ψ–insertion at the j-th marked point, and $[pt] \in H^2(E)$ is the Poincare-dual of a point.

A.1. Genus 1

In genus 1, the generating series are given by,

$$F_1^{K_X} := \left(\frac{1-\delta_{(E\cdot E),0}\chi(X)}{(E\cdot E)24} - \frac{1}{24}\right)\log Q^E + \sum_{\beta|\beta\cdot E>0} N_1(K_X,\beta)Q^\beta$$

$$F_1^{X/E} := -\frac{1-\delta_{(E\cdot E),0}\chi(X)}{(E\cdot E)24}\log Q^E + \sum_{\beta|\beta\cdot E>0}\frac{(-1)^{\beta\cdot E}}{\beta\cdot E}R_1(X(\log D),\beta)Q^\beta$$

$$F_{1,0^n}^{E} := \delta_{n,0}\frac{-1}{24}\log\left((-1)^{E\cdot E}\tilde{\mathcal{Q}}\right) + \sum_{d\geq 0}\tilde{\mathcal{Q}}^d\int_{[\overline{\mathcal{M}}_{1,n}(E,d)]^{vir}}\prod_{i=1}^{n} ev_i^*([pt])$$

Remark 23. *The closed string symplectic parameter Q keeps track of effective curve classes $\beta \in H_2^+(X,\mathbb{Z})$. It is related by mirror symmetry to the closed string complex parameter q on the stringy Kähler moduli space associated to K_X. The variable $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}(q)$ is related to Q by the change of variables,*

(A.1)

$$\tilde{\mathcal{Q}} = (-1)^{E\cdot E}Q^E\exp\left(\sum_{\beta|\beta\cdot E>0}(-1)^{\beta\cdot E}(\beta\cdot E)R_0(X(\log E),\beta)Q^\beta\right) = (-1)^{E\cdot E}\exp\left(-D^2F_0^{K_X}\right)$$

and can be expressed by Givental I-functions. For example, when K_X is local $\mathbb{P}^2$, the mirror geometry is a family of elliptic curves over a modular curve $Y_1(3) = \{q \in \mathbb{C}| q \neq \frac{-1}{27}, 0\} \cup \{\infty\}$, and $\tilde{\mathcal{Q}} = e^{3\left(\pi i + \frac{I_{12}(q)}{I_{11}(q)}\right)}$, where I_{11} and I_{12} are expressed in terms of Givental I-functions.

In Corollary 4, when $h = 1$, we have,

$$F_1^{K_X} = -F_1^{X/E} + \sum_{n\geq 0} F^E_{1,0^n}$$

The virtual dimension of $\overline{\mathcal{M}}_{1,n}(E,d)$ is n, hence,

$$F_1^{K_X} = -F_1^{X/E} + F^E_{1,\varnothing}$$

where $F^E_{1,0^n} := -\frac{1}{24}\log((-1)^{E\cdot E}\tilde{\mathcal{Q}}) + \sum_{d\geq 0}\tilde{\mathcal{Q}}^d\int_{\overline{\mathcal{M}}_{1,0}(E,d)} 1$. The invariants $F^E_{1,0^n}$ are computed in [**Dji**] and are given by,

$$\begin{aligned}\sum_{d\geq 0}\tilde{\mathcal{Q}}^d\int_{\overline{\mathcal{M}}_{1,0}(E,d)} 1 &= -\sum_{n\geq 1}\log(1-\tilde{\mathcal{Q}}^n)\\ &= \sum_{n\geq 1}\sum_{k\geq 1}\frac{\tilde{\mathcal{Q}}^{nk}}{k}\\ &= \sum_{n\geq 1}\left(\sum_{k|n,k\geq 1}\frac{1}{k}\right)\tilde{\mathcal{Q}}^n\end{aligned}$$

Remark 24. *In Equation A.1 defining the change of variables $\tilde{\mathcal{Q}} \leftrightarrow Q$, the invariants in the generating series $F^E_{g,\mathbf{a}}$ may initially seem to depend on the ambient surface $E\subset X$. However, that is the case after applying the mirror map $\tilde{\mathcal{Q}} \leftrightarrow Q$ to find the contributions to the local invariants of X. The generating series $F^E_{g,\mathbf{a}}$ viewed with $\tilde{\mathcal{Q}}$ as a formal variable does not depend on an embedding of $E\subset X$.*

The $g = 1$ log-local principle is,

$$
\begin{aligned}
F_1^{K_X} &= -F_1^{X/E} - \frac{1}{24}\log\left((-1)^{E\cdot E}\tilde{\mathcal{Q}}\right) + \sum_{n\geq 1}\left(\sum_{k|n,k\geq 1}\frac{1}{k}\right)\tilde{\mathcal{Q}}^n \\
&= -F_1^{X/E} - \frac{1}{24}\log(-1)^{E\cdot E}Q^E + \frac{1}{24}\sum_{\beta|\beta\cdot E>0}(-1)^{\beta\cdot E+1}(\beta\cdot E)R_0(X(\log E),\beta)Q^\beta \\
&+ \sum_{n\geq 1}\left(\sum_{j|n,j\geq 1}\frac{1}{j}\right)Q^{nE}\exp\left(n\sum_{\beta'|\beta'\cdot E>0}(-1)^{\beta'\cdot E}(\beta'\cdot E)R_0(X(\log E),\beta')Q^{\beta'}\right)
\end{aligned}
$$

where we changed variables $\tilde{\mathcal{Q}} \leftrightarrow Q$ in the second equality.

Let $[\bullet]_{Q^\beta}$ return the coefficient of Q^β for an expression $\bullet$.

Definition 38. *Define,*

$$
\delta_1(\beta) := \left[\sum_{n\geq 1}\left(\sum_{j|n,j\geq 1}\frac{1}{j}\right)Q^{nE}\exp\left[n\sum_{\beta'|\beta'\cdot E>0}(-1)^{\beta'\cdot E}(\beta'\cdot E)R_0(X(\log E),\beta')Q^{\beta'}\right] - \frac{\log(-1)^{E\cdot E}Q^E}{24}\right]_{Q^\beta}
$$

Thus, $\delta_1(\beta)$ captures the contribution from the stationary theory of E to the local invariants of K_X in class β .

Theorem 26 (Genus 1 log-local principle)**.**

$$
\begin{aligned}
N_1(K_X,\beta) = \frac{(-1)^{\beta\cdot E+1}}{\beta\cdot E}R_{1,(\beta\cdot E)}(X,\beta) - \frac{1}{24}\left((-1)^{E\cdot E}-1\right)\log Q^E \\
+ \frac{1}{24}(-1)^{\beta\cdot E+1}(\beta\cdot E)R_0(X(\log E),\beta) + \delta_1(\beta)
\end{aligned}
$$

A.1.1. Computing $\delta_1(\pi^* dH - C)$ when $X = \mathbb{P}^2$

Let $\beta = dH \in H_2(\mathbb{P}^2, \mathbb{Z})$ for $d > 0$. We specialize the genus 1 log-local principle to the Hirzebruch surface $\pi : (\mathbb{F}_1, \pi^* E - C) \to (\mathbb{P}^2, E)$, and determine the expressions for $R_{1,(\beta\cdot E-1)}(\mathbb{F}_1, \pi^*\beta - C)$. Theorem 26 gives us,

$$N^{K_{\mathbb{F}_1}}_{1,\pi^*\beta-C} = \frac{(-1)^{\beta\cdot E}}{\beta\cdot E-1} R_{1,(\beta\cdot E-1)}(\mathbb{F}_1, \pi^*\beta - C) + \frac{1}{24}(-1)^{\beta\cdot E}(\beta\cdot E - 1)R_{0,(\beta\cdot E-1)}(\mathbb{F}_1, \pi^*\beta - C)$$
$$+ \delta_1(\pi^*\beta - C)$$

Define $\tilde{A}_n$ to be the term in $\delta_1(\pi^*\beta - C)$ given by,

$$\tilde{A}_n := \exp[n \sum_{\beta' | \beta'\cdot(\pi^*E-C)>0} (-1)^{\beta'\cdot(\pi^*E-C)}(\beta'\cdot(\pi^*E-C))R_{0,(\beta'\cdot(\pi^*E-C))}(\mathbb{F}_1, \beta')Q^{\beta'}]$$

For $n \geq 1$, define $\tilde{\beta}_n := (\pi^*\beta - C) - nE = (d-3n)\pi^*H + (n-1)C$. We will require that $\tilde{\beta}_n \cdot (\pi^*E - C) > 0$, which implies $3d > 1 + 8n$. Since $n \geq 1$, this requires $d > 3$. We will address the case $d \leq 3$ separately.

Assume that $d > 3$. The coefficient of $Q^{\pi^*\beta - C}$ in A will contribute to the log invariant $R_{1,(\beta\cdot E-1)}(\mathbb{F}_1, \pi^*\beta - C)$. Equivalently, we will look for the ways $\tilde{\beta}_n$ appears in $\tilde{A}_n$ for $3d > 1 + 8n$.

We have,

$$\tilde{A}_n = [1 + n \sum_{\beta' | \beta' \cdot (\pi^* E - C) > 0} (-1)^{\beta' \cdot (\pi^* E - C)} (\beta' \cdot (\pi^* E - C)) R_{0,(\beta' \cdot (\pi^* E - C))}(\mathbb{F}_1, \beta') Q^{\beta'}$$
$$+ \frac{n^2}{2} (\sum_{\beta' | \beta' \cdot (\pi^* E - C) > 0} (-1)^{\beta' \cdot (\pi^* E - C)} (\beta' \cdot (\pi^* E - C)) R_{0,(\beta' \cdot (\pi^* E - C))}(\mathbb{F}_1, \beta') Q^{\beta'})^2 + \ldots]$$

Define T to be the cone of curve classes $\{\beta' | \beta' \cdot (\pi^* E - C) > 0\}$. If we write $\beta' = a\pi^* H + bC$, then T is the cone $\{a\pi^* H + bC | 3a + b > 0\}$. For $m \geq 1$, define $T^m := \{\beta_1 + \ldots + \beta_m | \beta_i \in T\}$. Notice that $T^m \subset \ldots \subset T^2 \subset T$. The cone T^m contains curve classes which can be reduced to a sum of m curve classes.

Let $1 \leq m < \infty$ be the highest natural number such that $\tilde{\beta}_n \in T^m$. Clearly, $\tilde{\beta}_n \in T^k$ for all $k \leq m$. Suppose that $\tilde{\beta}_n = \beta_n^{l1} + \ldots + \beta_n^{lk}$ is one of $1 \leq l \leq \ell_k$ possible decompositions of $\tilde{\beta}_n$ into *distinct* curve classes of T^k. We note that ℓ_k may equal ∞ (see Remark 6.2). The contribution of this decomposition is,

$$(\sum_{j | n, j \geq 1} \frac{1}{j}) \frac{n^k}{k!} (\prod_{i=1}^{k} (-1)^{\beta_n^{li} \cdot (\pi^* E - C)} (\beta_n^{li} \cdot (\pi^* E - C)) R_{0,(\beta_n^{li} \cdot (\pi^* E - C))}(\mathbb{F}_1, \beta_n^{li}))$$

Summing over all $n \geq 1$ such that $3d > 1 + 8n$, all cones T^k, and all ℓ_k decompositions, this is the total contribution from $\delta_1(\pi^* dH - C)$, hence,

$$\delta_1(\pi^* dH - C) := \sum_{\substack{n \geq 1 \\ 3d > 1 + 8n}} (\sum_{j | n, j \geq 1} \frac{1}{j}) \sum_{k=1}^{m} (\frac{n^k}{k!} \sum_{l=1}^{l_k} \prod_{i=1}^{k} (-1)^{\beta_n^{li} \cdot (\pi^* E - C)} (\beta_n^{li} \cdot (\pi^* E - C)) R_{0,(\beta_n^{li} \cdot (\pi^* E - C))}(\mathbb{F}_1, \beta_n^{li}))$$

When $d = 1, 2$, it is clear that there will be no contribution by definition of $\tilde{\beta}_n$ to $R_{1,(3d-1)}(\mathbb{F}_1, \pi^* dH - C)$, so we define $\delta_1(\pi^* dH - C) = 0$. When $d = 3$, there will be a contribution of +1 from the elliptic curve Q^{π^*3H-C} when $n = 1$. Thus, we define $\delta_1(\pi^*3H - C) = 1$.

Let's explicitly determine $\delta_1(\pi^*4H - C)$. In this case, we have $\tilde{\beta}_n = \pi^*H$. From T^1, there will be a contribution of $-3R_{0,(3)}(\mathbb{F}_1, \pi^*H)$ to the overall Gromov-Witten invariant. In T^2, π^*H can be decomposed as $\pi^*H = (k\pi^*H - (3k-1)C) + ((-k+1)\pi^*H + (3k-1)C)$ or $k\pi^*H - (3k-2)C + (-k+1)\pi^*H + (3k-2)C$. Hence, the contribution from these decompositions is,

$$-\sum_{k=1}^{\infty}[R_{0,(1)}(\mathbb{F}_1, k\pi^*H - (3k-1)C)R_{0,(2)}(\mathbb{F}_1, (-k+1)\pi^*H + (3k-1)C)$$
$$+ R_{0,(1)}(\mathbb{F}_1, (-k+1)\pi^*H + (3k-2)C)R_{0,(2)}(\mathbb{F}_1, k\pi^*H - (3k-2)C)]$$

There will be no contributions from T^k for $k > 2$. We switch to the basis $C = \pi^*H - F$ and F, where F denotes a fiber class. Explicit values for local invariants of $\mathbb{F}_1$ have been computed in Table 1 of [**LLW**]. We see that the only terms that turn up to be nonzero are $-[R_{0,(1)}(\mathbb{F}_1, C)R_{0,(2)}(\mathbb{F}_1, F) + R_{0,(1)}(\mathbb{F}_1, C)R_{0,(2)}(\mathbb{F}_1, F)] = -2R_{0,(1)}(\mathbb{F}_1, C)R_{0,(2)}(\mathbb{F}_1, F)$ and therefore,

$$\Delta(4) = -3R_{0,(3)}(\mathbb{F}_1, \pi^*H) - 2R_{0,(1)}(\mathbb{F}_1, C)R_{0,(2)}(\mathbb{F}_1, F)$$

Applying the genus 0 log-local principle, we have that

$$R_{0,(1)}(\mathbb{F}_1, C) = N_0(K_{\mathbb{F}_1}, C) = 1$$

$$R_{0,(2)}(\mathbb{F}_1, F) = -2N_0(K_{\mathbb{F}_1}, F) = 4$$

$$R_{0,(3)}(\mathbb{F}_1, \pi^*H) = 3N_0(K_{\mathbb{F}_1}, \pi^*H) = 9$$

Thus, $\Delta(4) = -35$.

In conclusion, we have the following table for the maximal tangency invariants in degrees $d = 1, 2, 3, 4$.

d	$R_{1,(3d-1)}(\mathbb{F}_1, \pi^* dH - C)$
1	$-2N^{K_{\mathbb{F}_1}}_{1,\beta} + \frac{1}{3}N^{K_{\mathbb{F}_1}}_{0,\beta}$
2	$5N^{K_{\mathbb{F}_1}}_{1,\beta} - \frac{125}{24}N^{K_{\mathbb{F}_1}}_{0,\beta}$
3	$-8N^{K_{\mathbb{F}_1}}_{1,\beta} - \frac{8}{3}R^{\mathbb{F}_1/E}_{0,\beta} + 8$
4	$11N^{K_{\mathbb{F}_1}}_{1,\beta} - \frac{121}{24}R^{\mathbb{F}_1/E}_{0,\beta} + 33R^{\mathbb{F}_1/E}_{0,\pi^*H} + 22R^{\mathbb{F}_1/E}_{0,C}R^{\mathbb{F}_1/E}_{0,F}$

A.2. Genus 2

We have the following generating series of local and relative invariants in genus 2,

$$F_2^{K_X} := \sum_{\beta|\beta\cdot E>0} N^{K_X}_{2,\beta} Q^\beta$$

$$F_2^{X/E} := \sum_{\beta|\beta\cdot E>0} \frac{(-1)^{\beta\cdot E+1}}{\beta\cdot E} R_2(X/E,\beta)Q^\beta$$

The operator D acts on the monoid ring of effective classes by $DQ^\beta = (\beta\cdot E)Q^\beta$, and $D\log Q^\beta := \frac{DQ^\beta}{Q^\beta} = \beta\cdot E$.

Theorem 27 (Genus 2 log-local principle)**.**

$$F_2^{K_X} = F_2^{X/E} + F^E_{1,(0)}D^2F_1^{X/E} + F^E_{2,(2)}D^4F_0^{X/E} - \frac{1}{2}(D^3F_0^{X/E})^2F^E_{2,(1,1)}$$

It was shown by [**OP**] that the generating series $F^E_{h,\mathbf{a}}$ can be expressed in terms of Eisenstein series E_{2k} of weight $2k$, i.e.

$$F^E_{h,\mathbf{a}} \in \mathbb{Q}[E_2, E_4, E_6]$$

Explicitly, we have that

$$F^E_{1,(0)}(\tilde{\mathcal{Q}}) := \frac{-E_2}{24}$$
$$F^E_{2,(2)}(\tilde{\mathcal{Q}}) := \frac{1}{5760}(2E_4 + 5E_2^2)$$
$$F^E_{2,(1,1)}(\tilde{\mathcal{Q}}) := \frac{-1}{25920}(2E_6 + 3E_2E_4 - 5E_2^3)$$

where

$$E_{2k}(\tilde{\mathcal{Q}}) = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\frac{n^{2k-1}\tilde{\mathcal{Q}}^n}{1-\tilde{\mathcal{Q}}^n}$$

with the Bernoulli numbers B_{2k} defined by $\frac{t}{e^t-1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}$. The E_{2k} are quasimodular forms of weight $2k$ for the modular group $SL_2(\mathbb{Z})$.

A.3. Evaluation of Vertex V_3 in genus 1 in Chapter 5

We directly evaluate the genus-1 invariant associated to vertex V_3 in the degeneration formula,

$$\int_{[\overline{\mathcal{M}}_{1,2}(\mathbb{F}_1(\log F),\pi^*H)]^{vir}} -\lambda_1 ev_1^*([pt_1])ev_2^*([pt_2])$$

where $[pt_1] \in A^2(\mathbb{F}_1)$ is the Poincare dual of a point in the interior and $[pt_2] \in A^1(\mathbb{P}^1)$ is the Poincare dual of a point on a fiber class of $\mathbb{F}_1$. Recall that on $\overline{\mathcal{M}}_{1,1}$, we have,

$$\lambda_1 = \frac{1}{12}\delta_0$$

where $\delta_0 \in A^1(\overline{\mathcal{M}}_{1,1})$ is the class of a point. We take for representative of δ_0 the point corresponding to the nodal rational cubic, and resolve the node. The integral becomes,

$$\left(\frac{1}{2}\right)\left(\frac{-1}{12}\right)\int_{[\overline{\mathcal{M}}_{0,4}(\mathbb{F}_1(\log\mathbb{P}^1),\pi^*H)]^{vir}} ev_1^*([pt_1])ev_2^*([pt_2])(ev_3\times ev_4)^*(D\times D)$$

where the $\frac{1}{2}$ comes from the two ways of labelling the two marked points that resolve the node. The class $D\times D$ is the diagonal curve class in $A^2(\mathbb{F}_1\times\mathbb{F}_1)$, which is

$$D\times D=(1\times pt)+(pt\times 1)+(\pi^*H\times\pi^*H)+(C\times C)$$

The first two terms in $D\times D$ will contribute zero, by the Fundamental Class Axiom. The last term will also contribute zero by the Divisor Axiom, since $\pi^*H\cdot C=0$. Hence, the integral becomes

$$\frac{-1}{24}\int_{[\overline{\mathcal{M}}_{0,4}(\mathbb{F}_1(\log\mathbb{P}^1),\pi^*H)]^{vir}} ev_1^*([pt_1])ev_2^*([pt_2])ev_3^*(\pi^*H)ev_4^*(\pi^*H)$$

Using the Divisor Axiom again, this is,

$$\frac{-1}{24}\int_{[\overline{\mathcal{M}}_{0,2}(\mathbb{F}_1(\log\mathbb{P}^1),\pi^*H)]^{vir}} ev_1^*([pt_1])ev_2^*([pt_2])$$

This invariant is the number of lines through two points, and hence the above evaluates to $\frac{-1}{24}$.