Every smooth cubic surface has exactly 27 lines (GSS talk)

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1 Introduction

I'll present two proofs of the classical fact in enumerative geometry that every smooth cubic surface $\subseteq \mathbb{P}^3$ has *exactly* 27 lines. There is a "simpler" proof that uses mainly linear algebra to first show there are exactly 27 lines on the Fermat cubic $x_0^3 + x_1^3 + x_2^3 + x_4^3 = 0$, and then proves every other smooth cubic must have the same number of lines. I'll present two other proofs: 1) Realizing the smooth cubic surface as the blow up of $\mathbb{CP}^2$ at 6 points, and counting lines in the blow up 2) Schubert calculus on the cohomology of $Gr(2,4)$, the Grassmannian of 2-planes in $\mathbb{C}^4$.

2 Counting lines on smooth cubic surface as $Bl_{\mathbf{6pts}}\mathbb{CP}^2$

Take 6 sufficiently general points in $\mathbb{CP}^2$, i.e. no three points are collinear, no six lie on a conic. There exists a rational map $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^3$ as follows: Take the vector space of cubic polynomials in $\mathbb{CP}^2$, which is $\binom{3+2}{2} = 10$ dimensional. Asking that the cubic passes through 6 points imposes 6 linearly independent conditions. Hence, there exist 4 linearly independent cubics f_i vanishing at the 6 points. Define

$$\varphi([x_0 : x_1 : x_2]) := [f_0(x_0, x_1, x_2) : f_1(x_0, x_1, x_2) : f_2(x_0, x_1, x_2) : f_3(x_0, x_1, x_2)]$$

This map is clearly undefined at the six points. After blowing up at the 6 points, by the theorem of elimination of indeterminacy, there exists a morphism $\varphi' : Bl_{6pts}\mathbb{CP}^2 \to \mathbb{CP}^3$ and the commutative diagram.

$$\begin{tikzcd}
Bl_{6pts}\mathbb{CP}^2 \arrow[r, "\pi"] \arrow[dr, "\varphi'"'] & \mathbb{CP}^2 \arrow[d, dashed, "\varphi"] \\
& \mathbb{CP}^3
\end{tikzcd}$$

, where π is the contraction map (Draw picture of blow up). The map φ' is an embedding and is the same embedding from the Kodaira embedding theorem with the anti-canonical bundle $\mathcal{O}(3L - E_1 - \ldots - E_6)$ of $Bl_{6pts}\mathbb{CP}^2$. Hence, the image $S := \varphi'(Bl_{6pts}\mathbb{CP}^2)$ is a smooth surface. We need to verify it's cubic; it's enough to check $\int_S H^2 = 3$, since by Bezout's theorem, a cubic surface will intersect a line at 3 points ($H = PD$(hyperplane), so H^2 is the class of intersection of two hyperplanes or a line). The embedding map satisfies $c_1(\varphi')^*(\mathcal{O}(1)) = (\varphi')^*H = 3H - PD(E_1) - \ldots - PD(E_6) = c_1(\mathcal{O}(3L - E_1 - \ldots - E_6))$ So, since cup product is Poincare dual to intersection,

$$\int_S H^2 = \int_{Bl_{6pts}\mathbb{CP}^2} ((\varphi')^*H)^2 = \int_{Bl_{6pts}\mathbb{CP}^2} (3H - PD(E_1) - \ldots - PD(E_6))^2 = (3L - E_1 - \ldots - E_6) \cdot (3L - E_1 - \ldots - E_6)$$

We have that L is the class of line that doesn't intersect E_i, $E_i^2 = -1$, and $E_i \cdot E_j = 0$ since the exceptional divisors are disjoint from each other. Thus, the above integral is 3 and the surface is cubic.

We can characterize lines on S as rational curves with self-intersection -1.

Lemma 1 *If ℓ is a line on S, then $\ell^2 = -1$ on S. Conversely, if $C \subset S$ is a smooth irreducible rational curve with $C^2 = -1$, then C is a line.*

Proof: Let $\ell \subset S$ be a line. By the adjunction formula $(2g - 2 = \ell \cdot (\ell + K_S))$, we have $-2 = \ell^2 + \ell \cdot (-L)$, since the canonical divisor $K_S = -L$. Hence $\ell^2 = -1$ on S. Conversely, by adjunction again, $-2 = C^2 + C \cdot K_S$, and thus $1 = C \cdot L$. The degree of C is 1, so C is a line.

Thus, lines on S are smooth rational curves with self intersection -1. So to count the lines on S, we count rational curves with self intersection -1. Using the blow up description of the cubic surface, we see that the exceptional curves $E_1, \ldots, E_6$ give us 6 lines. Furthermore, the proper transform $L - E_i - E_j$ of the unique line through p_i and p_j in $\mathbb{CP}^2$ give us $\binom{6}{2} = 15$ lines, as $(L - E_i - E_j)^2 = -1$. The proper transform $2L - E_{i_1} - \ldots - E_{i_5}$ of the unique conic through 5 points gives us $\binom{6}{5} = 6$ lines (Draw pictures here). This already gives us a total of 27 lines! To see that these are the only lines, any other line in $Bl_{6pts}\mathbb{CP}^2$ is of the form $D = aL - b_1E_1 - \ldots - b_6E_6$ with $a > 0, b_i \geq 0$ and $D^2 = -1$. It follows from Cauchy-Schwarz implying $a = 1$ or 2, and $a^2 - \sum b_i^2 = -1$ that the above are the only lines.

We have shown that when the cubic surface is a blow up of the projective plane at 6 sufficiently general points, then it has exactly 27 lines. But actually it's a theorem that every smooth cubic surface is obtained in this way. Thus every smooth cubic surface has exactly 27 lines (Heuristic: we know $\{Bl_{6pts}\mathbb{CP}^2\} \subseteq$ {smooth cubic surfaces}. Both sides are 19 dimensional, LHS forms a open dense subset).

3 Schubert calculus on $Gr(2,4)$

Every 2-plane in $\mathbb{C}^4$ may be represented as a rank 2, 2×4 matrix with complex coefficients. We are interested in $Gr(2,4)$, because 2-planes in $\mathbb{C}^4$ correspond to lines in $\mathbb{P}^3$, after projectivization of the row space. After Gaussian elimination, there are 6 possible row echelon forms of an element in $Gr(2,4)$,

$$\Sigma_{0,0} = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix}, \Sigma_{1,0} = \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \Sigma_{2,0} = \begin{bmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Sigma_{1,1} = \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}, \Sigma_{2,1} = \begin{bmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \Sigma_{2,2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

, where $*$ signifies an arbitrary complex number. We see that $\Sigma_{0,0}$ is a complex 4-cell, etc. and they provide a cellular decomposition of $Gr(2,4)$. These are called the *Schubert cells*. We first find a geometric interpretation of the cellular decomposition. Fix a flag $p \in L \subset H$, where we take the point $p = [0:0:0:1]$, the line $L = \{x_0 = x_1 = 0\}$, and the hyperplane $\{x_0 = 0\}$. Using the matrix descriptions of the $\Sigma_{i,j}$, it is easy to see that

$$\Sigma_{2,2} = \{L\}, \Sigma_{2,1} = \{\ell | p \in \ell \subset H\}, \Sigma_{1,1} = \{\ell | \ell \subset H\}, \Sigma_{2,0} = \{\ell | p \in \ell\}, \Sigma_{1,0} = \{\ell | \ell \cap L \neq \varnothing\}, \Sigma_{0,0} = Gr(2,4)$$

and we have the cellular decomposition,

$$\Sigma_{2,2} \subset \Sigma_{2,1} \subset \Sigma_{1,1} \cup \Sigma_{2,0} \subset \Sigma_{1,0} \subset \Sigma_{0,0}$$

Since these are complex cells, the cohomology is even dimensional and free abelian, i.e.

$$H^0(Gr(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_{0,0}, H^2(Gr(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_{1,0}$$

$$H^4(Gr(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_{2,0} \oplus \mathbb{Z}\sigma_{1,1}, H^6(Gr(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_{2,1}, H^8(Gr(2,4);\mathbb{Z}) = \mathbb{Z}\sigma_{2,2}$$

, where $\sigma_{i,j}$ are the generators of cohomology corresponding to the $\Sigma_{i,j}$.

The calculation of 27 lines on the cubic surface will be a characteristic class calculation involving the Schubert classes. So, we calculate part of the cohomology ring of $Gr(2,4)$, namely the classes $\sigma_{11}\sigma_{20}, \sigma_{10}^2$, and σ_{11}^2. First, fix another flag $p' \in L' \subset H'$, in order the make intersections transverse. We see that $\sigma_{11}\sigma_{20}$ represent the lines ℓ which are contained in H and meet p'. Generically $p' \notin H$, so there are no such lines, i.e. $\sigma_{11}\sigma_{20} = 0$. We see that σ_{11}^2 represent the lines ℓ contained in the hyperplanes H and H'. The intersection $H \cap H'$ will be only one line, so $\sigma_{11}^2 = \sigma_{22}$. To calculate σ_{10}^2, we know that $\sigma_{10}^2 = a\sigma_{20} + b\sigma_{11}$ for some integers a, b. From this, we have $\sigma_{10}^2\sigma_{20} = a\sigma_{22}$ and $\sigma_{10}^2\sigma_{11} = b\sigma_{11}$. Taking a third flag $p'' \in L'' \subset H''$, the class $\sigma_{10}^2\sigma_{20}$ represents the lines ℓ that meet L, L' and contain p''. Lines meeting L' and containing p'' span a plane. The line L will intersect the plane at only one point, so there is only one line satisfying all three conditions. Therefore, $\sigma_{10}^2\sigma_{20} = \sigma_{22}$. Similarly, $\sigma_{10}^2\sigma_{11}$ will represent lines ℓ intersecting L, L' and contained in the hyperplane H''. Each line L, L' will intersect H'' at one point respectively, and there exists a unique line connecting the two points. Thus $a = b = 1$, i.e . $\sigma_{10}^2 = \sigma_{20} + \sigma_{11}$.

Recall that we have the tautological bundle E over $Gr(2,4)$. We will calculate the classes of $Sym^3 E^*$, the fiber of which are degree 3 homogeneous forms on the line ℓ. Now take a smooth cubic surface $S = \{F = 0\} \subseteq \mathbb{P}^3$, which is the zero-set of a degree 3 homogeneous polynomial F. Given $\ell \subseteq \mathbb{CP}^3$, we can restrict $F|_\ell$, and thus get a section s of $Sym^3 E^*$. The top Chern class of $Sym^3 E^*$ represents the 0-set of a generic section. The zeros of the section are the lines contained in the cubic surface S! Thus we calculate

$$\int_{Gr(2,4} c_4(Sym^3 E^*)$$

with $rk_{\mathbb{C}} Sym^3 E^* = \binom{2+3-1}{1} = 4$. The splitting principle can be used to calculate $c_4(Sym^3 E^*)$: any formula of Chern classes of a vector bundle derived from assuming E is a direct sum of line bundles actually holds. Using this, we have

$$c_4(Sym^3 E^*) = 9c_2(E^*)(2c_1(E^*)^2 + c_2(E^*)), \text{with } c_1(E^*) = \sigma_{10}, c_2(E^*) = \sigma_{11}$$

Thus, $c_4 = 9\sigma_{11}(2\sigma_{10}^2 + \sigma_{11}) = 27\sigma_{22} \Rightarrow \int_{Gr(2,4)} c_4 = 27$. This number is up to multiplicity, but if we assume the section s has transverse zeros, then there exactly 27 lines. The section is transverse precisely when X is a *smooth* cubic surface.

References

[1] Vakil's course on complex algebraic surfaces

[2] Jack Huizenga's quora post

[3] Hartshorne chapter 5

[4] Griffiths Harris chapter on surfaces