Point-like Bounding Chains in Open Gromov-Witten Theory

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- Define $GW_{A,k}(\alpha_1, \ldots, \alpha_k) := \int_{\overline{\mathcal{M}_{0,k}}(A,J)} ev_1^* \alpha_1 \wedge \ldots \wedge ev_k^* \alpha_k$, where $\overline{\mathcal{M}_{0,k}}(A, J)$ is the Gromov compactification of the moduli space of J-holomorphic spheres with k marked points, α_i are Poincaré dual to X_i, and $(ev_1, \ldots, ev_k)$ are evaluation maps of u at the k-marked points.

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- The integral needs to be made sense of, since $\overline{\mathcal{M}_{0,k}}(A, J)$ does not carry a fundamental class. (Methods such as pseudocycles, virtual fundamental classes, Kuranishi structures, polyfold theory, etc. have been used)
- If the boundary strata from the Gromov compactification have **codimension** $\geq$ **2**, then Gromov-Witten invariants can be defined. They depend neither on the almost complex structure J as long as it tames ω, nor on the representatives of the cohomology classes α_i.

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- Unlike in the closed case, there exist boundary strata of **codimension 1** in the moduli space of J-holomorphic discs.
- Intuitively, because of Stokes' theorem, you would not expect the integral above to be independent of the representatives of cohomology classes being integrated anymore.

Some previous work in defining Open Gromov Witten invariants

- Liu defined OGWs for (M, L) carrying an S^1 action (2002)
- Using anti-symplectic involution, Welschinger defined counts of real rational J-holomorphic curves in dimensions 2,3 (2005)
- Fukaya defined OGWs for Calabi-Yau 3-fold and Maslov 0 Lagrangian (2011)
- Georgieva extended Welschinger's work to higher, odd dimensions (2016)

Bounding cochains in Open Gromov Witten invariants

- Fukaya introduced bounding cochains to show Lagrangian Floer theory can be defined in more general settings.
- The bounding cochain deforms the Floer coboundary operator to one that squares to 0, and "cancels" codimension 1 bubbling.
- This presentation seeks to explain Solomon-Tukachinsky's approach of **defining OGWs using bounding cochains**.

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- Let $\Pi = H_2(X, L)/S_L$ where S_L is a subgroup of $\ker(\omega \oplus \mu) : H_2(X, L) \to \mathbb{R} \oplus \mathbb{Z}$. Denote $\beta_0 \in \Pi$ to be the zero element.

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- Define the Novikov field $\Lambda = \{\sum_{i=0}^{\infty} a_i T^{\beta_i} | a_i \in \mathbb{R}, \beta_i \in \Pi, \omega(\beta_i) \geq 0, \lim_{i \to \infty} \omega(\beta_i) = \infty\}$ and $\Lambda^+ = \{\sum_{i=0}^{\infty} a_i T^{\beta_i} \in \Lambda | \omega(\beta_i) > 0\}$.

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- Denote cochains on L by $A^*(L)$, and cochains on X relative to L by $A^*(X, L)$.
- Introduce formal variables $s, t_0, \ldots, t_N$.

Moduli spaces involved

- **Gromov Compactness** states that for a sequence of J-holomorphic discs with uniformly bounded energy, there exists a subsequence that converges up to $PSL_2(\mathbb{R})$ action to an at worst nodal J-holomorphic disc with components that are discs or spheres.

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- Denote $\mathcal{M}_{k+1,l}(\beta)$ to be the moduli space of $g = 0$, J-holomorphic stable maps $u : (\Sigma, \partial\Sigma) \to (M, L)$ with 1 boundary component, $k+1$ boundary marked points, and l interior marked points. Let $evb_j : \mathcal{M}_{k+1,l}(\beta) \to L$ be the evaluation map at the b_j boundary marked point, and $evi_j : \mathcal{M}_{k+1,l}(\beta) \to M$ be the evaluation map at the i_j interior marked point.

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- Assume $\mathcal{M}_{k+1,l}(\beta)$ is a **smooth orbifold with corners**, and the evaluation maps are **proper submersions**. The latter assumption will allow us to define pushforward operations with the evaluation map. This holds for $(\mathbb{CP}^n, \mathbb{RP}^n)$.

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- The relative spin condition on L makes $\mathcal{M}_{k+1,l}(\beta)$ orientable.

A^∞ algebra associated to cochains of L

- Let $R := \Lambda[[s, t_0, \ldots, t_N]]$ and $Q := \mathbb{R}[t_0, \ldots, t_N]$. Take the ideals $\mathcal{I}_R := \langle s, t_0, \ldots, t_N \rangle \lhd R$ and $\mathcal{I}_Q := \langle t_0, \ldots, t_N \rangle \lhd Q$.

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- Let $C := A^*(L) \otimes R$ and $D := A^*(X, L) \otimes Q$. Choose $\gamma \in \mathcal{I}_Q A^*(X, L; Q)$ with $d\gamma = 0, \deg \gamma = 2$. Define the A^∞ structure maps $\mathfrak{m}_k^\gamma : C^{\otimes k} \to C$ for $k \geq 0$ by

$$\mathfrak{m}_k^\gamma(\alpha_1, \ldots, \alpha_k) := \delta_{k,1} d\alpha_1 + \sum_{\beta \in \Pi, l \geq 0} \frac{T^\beta}{l!} (evb_0)_* \Big(\bigwedge_{j=1}^{l} (evi_j^\beta)^* \gamma \wedge \bigwedge_{j=1}^{k} (evb_j^\beta)^* \alpha_j \Big)$$

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- $(evb_0)_*$ is defined by **integration over the fiber**, as it's a proper submersion. The output is a chain given by all points that lie on a boundary of a disc with boundary constraints $\alpha_1, \ldots, \alpha_k$ and l interior constraints γ, for all l.

A^∞ algebra associated to cochains of L, cont'd

- The $\{\mathfrak{m}_k\}_{k=0}^{\infty}$ satisfy the A^∞ **relations**, i.e.

$$\sum_{k_1+k_2=k+1, 1\leq i\leq k_1} (-1)^{\epsilon(\alpha)} \mathfrak{m}_{k_1}^{\gamma}(\alpha_1, \ldots, \alpha_{i-1}, \mathfrak{m}_{k_2}^{\gamma}(\alpha_i, \ldots, \alpha_{i+k_2-1}), \alpha_{i+k_2}, \ldots, \alpha_k) = 0$$

- Furthermore, define $\mathfrak{m}_{-1}^{\gamma} := \sum_{\beta\in\Pi, l\geq 0} \frac{T^\beta}{l!} \int_{\mathcal{M}_{0,l}(\beta)} \bigwedge_{i=1}^{l} (evi_j^\beta)^* \gamma$

Bounding pairs and the superpotential

- Define (γ, b) to be a **bounding pair** with $\gamma \in \mathcal{I}_Q A^*(X, L; Q), d\gamma = 0, \deg \gamma = 2$, and $b \in I_R C, \deg_C b = 1$ if the **Maurer-Cartan equation** holds

$$\sum_{k \geq 0} \mathfrak{m}_k^{\gamma}(b^{\otimes k}) = c \cdot 1$$

where 1 is the constant function on L and $c \in \mathcal{I}_R$ with deg $c = 2$. Here b is called a **(weakly) bounding cochain**.

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- Define the **superpotential**

$$\Omega(\gamma, b) = \Omega_J(\gamma, b) := (-1)^n (\sum_{k \geq 0} \frac{1}{k+1} \langle \mathfrak{m}_k^{\gamma}(b^{\otimes k}), b \rangle + \mathfrak{m}_{-1}^{\gamma})$$

Here $\langle \xi, \eta \rangle := (-1)^{|\eta|} \int_L \xi \wedge \eta$ is the Poincaré pairing.

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- The superpotential is a function on the space of bounding pairs. Note b is not necessarily closed.

Invariance of the superpotential

- Let (γ, b) be a bounding pair with respect to J, and (γ', b') a bounding pair with respect to J'. There exists an equivalence relation called **gauge equivalence** on bounding pairs that essentially constructs an isotopy between them.
- S-T showed the following,

Theorem (Invariance of the superpotential, S-T)

If $(\gamma, b) \sim (\gamma', b')$, *then* $\Omega_J(\gamma, b) = \Omega_{J'}(\gamma', b')$

- In proving this invariance, need to consider curve classes $\beta \in \mathrm{Im}\{(H_2(X, L) \to H_1(L))\}$ as a special case. "Cancel out" this possible degeneration by also considering moduli space of spheres with 1 marked point intersecting L.

Classification of Bounding Pairs

- Define a map $\rho : \{\text{bounding pairs}\}/\sim \to (\mathcal{I}_Q H^*(X, L; Q))_2 \oplus (\mathcal{I}_R)_{1-n}$,

$$\rho([\gamma, b]) := ([\gamma], \int_L b)$$

- In certain settings, ρ is bijective,

Theorem (Classification of bounding pairs, S-T)

Assume $H^*(L;\mathbb{R}) \cong H^*(S^n;\mathbb{R})$. Then ρ is bijective.

- Reason for the term "**point-like**": bounding cochain b is therefore determined up to gauge equivalence by its form part of degree n, which will be a multiple of the Poincaré dual of a point.

Definition of OGWs using bounding cochains

- Assuming the theorems above, we can define Open Gromov-Witten invariants of the pair (M, L). Take a basis $\Gamma_0, \ldots, \Gamma_N$ of $H^*(M, L; \mathbb{R})$. Set $\Gamma := \sum_{j=0}^{N} t_j \Gamma_j$ and $\deg t_j = 2 - |\Gamma_j|$.

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- Define the **Open Gromov-Witten Invariants** $OGW_{\beta,k} : H^*(M, L; \mathbb{R})^{\otimes l} \to \mathbb{R}$ by,

$$OGW_{\beta,k}(\Gamma_{i_1}, \ldots, \Gamma_{i_l}) := \text{coefficient of } T^\beta \text{ in } \partial_{t_{i_1}} \ldots \partial_{t_{i_l}} \partial_s^k \Omega(\gamma, b)|_{s=t_j=0}$$

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- The OGWs defined this way are invariant with respect to ω-tame almost complex structures and representatives of the cohomology class of interior constraints $[\gamma]$, because of gauge equivalence.

Axioms of OGWs

- Kontsevich-Manin (1994) gave axioms that a Gromov-Witten theory should satisfy.
- Solomon-Tukachinsky showed $OGW_{\beta,k}$ defined above satisfy some of the Kontsevich-Manin axioms, including
- (1) **Degree axiom**: $OGW_{\beta,k}(A_1, \ldots, A_l) = 0$ unless $n - 3 + \mu(\beta) + k + 2l = kn + \sum_{j=1}^{l} |A_j|$
- (2) **Fundamental class axiom**:
$$OGW_{\beta,k}(1, A_1, \ldots, A_{l-1}) = \begin{cases} -1 & \text{when } (k, l, \beta) = (1, 1, \beta_0) \\ 0 & \text{otherwise} \end{cases}$$
- (3) **Deformation invariance**: $OGW_{\beta,k}$ remain constant under deformations of the symplectic form ω, for which L remains Lagrangian.

Properties of $\mathfrak{q}_{k,l}$

- In the definition of

$$\mathfrak{m}_k^{\gamma}(\alpha_1,\ldots,\alpha_k) := \delta_{k,1} d\alpha_1 + \sum_{\beta\in\Pi, l\geq 0} \frac{T^{\beta}}{l!} (evb_0)_*\Big(\bigwedge_{j=1}^{l} (evi_j^{\beta})^*\gamma \wedge \bigwedge_{j=1}^{k} (evb_j^{\beta})^*\alpha_j\Big)$$

it is useful for calculational purposes to isolate terms in the sum and define

$$\mathfrak{q}_{k,l}^{\beta}(\alpha_1,\ldots,\alpha_k;\gamma_1,\ldots,\gamma_l) := (evb_0)_*\Big(\bigwedge_{j=1}^{l} (evi_j^{\beta})^*\gamma_j \wedge \bigwedge_{j=1}^{k} (evb_j^{\beta})^*\alpha_j\Big)$$

for $(k,l,\beta) \neq (1,0,\beta_0)$ and $\mathfrak{q}_{1,0}^{\beta_0}(\alpha) = d\alpha$, so that

$$\mathfrak{m}_k^{\gamma}(\alpha_1,\ldots,\alpha_k) := \sum_{\beta\in\Pi, l\geq 0} \frac{T^{\beta}}{l!} \mathfrak{q}_{k,l}^{\beta}(\alpha_1,\ldots,\alpha_k;\gamma,\ldots,\gamma)$$

Properties of $\mathfrak{q}_{k,l}$

- The operators $\mathfrak{q}^{\beta}_{k,l}$ satisfy the following properties:
- **Fundamental class**: $\mathfrak{q}^{\beta}_{k,l}(\alpha_1, \ldots, \alpha_k; 1, \gamma_1, \ldots, \gamma_{l-1}) = -1$ when $(k, l, \beta) = \begin{cases} -1 & \text{if } (0, 1, \beta_0) \\ 0 & \text{otherwise} \end{cases}$
- **Energy zero**: $\mathfrak{q}^{\beta_0}_{k,l}(\alpha_1, \ldots, \alpha_k; \gamma_1, \ldots, \gamma_l) = \begin{cases} d\alpha_1 & \text{if } (k, l) = (1, 0) \\ (-1)^{\deg \alpha_1} \alpha_1 \wedge \alpha_2 & \text{if } (k, l) = (2, 0) \\ -\gamma_1|_L & \text{if } (k, l) = (0, 1) \\ 0 & \text{otherwise} \end{cases}$
- **Top Degree**: Suppose $(k, l, \beta) \notin \{(1, 0, \beta_0), (0, 1, \beta_0), (2, 0, \beta_0)\}$, then $(\mathfrak{q}^{\beta}_{k,l}(\alpha; \gamma))_n = 0$ for all lists α, γ.

Gauge equivalence

- Work with a family of almost complex structures $\{J_t\}$ and a slightly bigger moduli space:

$$\widetilde{\mathcal{M}}_{k+1,l}(\beta) := \{(t, u, \overrightarrow{z}, \overrightarrow{w}) \,|(u, \overrightarrow{z}, \overrightarrow{w}) \in \widetilde{\mathcal{M}}_{k+1,l}(\beta; J_t)\}$$

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- Have evaluation maps $\widetilde{evb_j} : \widetilde{\mathcal{M}}_{k+1,l}(\beta) \to I \times L$, and $\widetilde{evi_j}$.

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- Have evaluation maps $\widetilde{evb_j} : \widetilde{\mathcal{M}}_{k+1,l}(\beta) \to I \times L$, and $\widetilde{evi_j}$.
- Can similarly define A^∞ structure maps $\widetilde{\mathfrak{m}}_k : A^*([0,1] \times L, \Lambda)^{\otimes k} \to A^*([0,1] \times L, \Lambda)$.

Gauge equivalence

- Work with a family of almost complex structures $\{J_t\}$ and a slightly bigger moduli space:

$$\widetilde{\mathcal{M}}_{k+1,l}(\beta) := \{(t, u, \vec{z}, \vec{w}) \mid (u, \vec{z}, \vec{w}) \in \widetilde{\mathcal{M}}_{k+1,l}(\beta; J_t)\}$$

- Have evaluation maps $\widetilde{evb_j} : \widetilde{\mathcal{M}}_{k+1,l}(\beta) \to I \times L$, and $\widetilde{evi_j}$.
- Can similarly define A^∞ structure maps $\widetilde{\mathfrak{m}}_k : A^*([0,1] \times L, \Lambda)^{\otimes k} \to A^*([0,1] \times L, \Lambda)$.
- A bounding pair $(\gamma, b)_J$ is **gauge equivalent** to a bounding pair $(\gamma', b')_{J'}$ if $\exists \widetilde{b} \in A^*([0,1] \times L; \Lambda)$ satisfying $\widetilde{b}|_{\{0\}\times L} = b$, $\widetilde{b}|_{\{1\}\times L} = b'$ and

$$\sum_{k \geq 0} \widetilde{\mathfrak{m}}_k(\widetilde{b}^{\otimes k}) = c \cdot 1$$

and $\exists \widetilde{\gamma} \in A^*([0,1] \times X, [0,1] \times L; \Lambda)$ with $d\widetilde{\gamma} = 0$ and $\widetilde{\gamma}|_{\{0\}\times X} = \gamma$, $\widetilde{\gamma}|_{\{1\}\times X} = \gamma'$

Proof of classification of bounding pairs (for cohomology spheres)

- We first show ρ is well defined: assuming $n > 0$, if $(\gamma, b) \sim (\gamma', b')$, then $[\gamma] = [\gamma']$ and $\int_L b = \int_{L'} b'$.
- Proof of well-definedness: By definition of gauge equivalence, there exist $\widetilde{\gamma} \in A^*([0,1] \times X, [0,1] \times L; \Lambda)$ with $d\widetilde{\gamma} = 0$ and $\widetilde{\gamma}|_{\{0\}\times X} = \gamma$, $\widetilde{\gamma}|_{\{1\}\times X} = \gamma'$. By a generalized Stokes' theorem on orbifolds with corners, $[\gamma] = [\gamma']$. We also have $\widetilde{b} \in A^*([0,1] \times L; \Lambda)$ with $\widetilde{b}|_{\{0\}\times L} = b$, $\widetilde{b}|_{\{1\}\times L} = b'$, and satisfying the Maurer-Cartan equation.

Proof of classification of bounding pairs (for cohomology spheres), cont'd

- We have

$$
\begin{aligned}
\int_L b' - \int_L b = \int_{\partial(I\times L)} \tilde{b} &= \int_{I\times L} d\tilde{b} \\
&= \int_{I\times L} (\tilde{c}\cdot 1 - \sum_{(k,l,\beta)\neq(1,0,\beta_0)} \tilde{\mathfrak{q}}_{k,l}(\tilde{b}^k;\tilde{\gamma}^k))_{n+1} \quad \text{(Maurer-Cartan)} \\
&= \int_{I\times L} (\tilde{c}\cdot 1)_{n+1} - (\tilde{\mathfrak{q}}_{2,0}(\tilde{b}^2) + \tilde{\mathfrak{q}}_{0,1}(\gamma))_{n+1} \quad \text{(Top Degree)} \\
&= \int_{I\times L} (\tilde{c}\cdot 1)_{n+1} - (\tilde{b}\wedge\tilde{b} - \tilde{\gamma}|_{I\times L})_{n+1}
\end{aligned}
$$

- This equals zero since $\deg \tilde{b} = 1$ and $\tilde{\gamma} \in A^*(I\times X, I\times L)$, $d\tilde{\gamma} = 0$. Since $\tilde{c} \in A^*([0,1];\Lambda)$ and $n > 0$, $(\tilde{c})_{n+1} = 0$. Thus, the map ρ is well defined.

Definition of the obstruction classes

- To prove classification or bijectivity of the map ρ, we define obstruction classes motivated by [FOOO]. The vanishing of obstruction classes signifies the existence of a bounding cochain.
- There exists a natural valuation $\nu : R := \Lambda[[s, t_0, \ldots, t_N]] \to \mathbb{R}_{\geq 0}$ defined by

$$\nu\left(\sum_{j=0}^{\infty} a_j T^{\beta_j} s^{k_j} \prod_{a=0}^{N} t_a^{l_{aj}}\right) := \inf_{j, a_j \neq 0} \left(\omega(\beta_j) + k_j + \sum_{a=0}^{N} l_{aj}\right)$$

- Denote $F^E R$ the filtration on R defined by $\lambda \in F^E R \iff \nu(\lambda) > E$. The filtration defines a topology on R: a sequence $\{x_i\}$ converges in R if $\forall E \in \mathbb{R}_{\geq 0}, \exists N$ such that for $\forall n \geq N$, $a_n \in F^E R$.

Definition of the obstruction classes

- Given $b \in C := A^*(L) \otimes \Lambda[[s, t_0, \dots, t_N]]$, write $b = \sum_{j=0}^{\infty} \lambda_j b_j$ with $b_j \in A^*(L), \lambda_j = T^{\beta_j} s^{k_j} \prod_{a=0}^{N} t_a^{l_{aj}}$. We can order the $\{\lambda_j\}_{j=0}^{\infty}$ such that if $i \le j$, then $\nu(\lambda_i) \le \nu(\lambda_j)$.

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- Define the index κ_l to be the largest index of $\{\lambda_j\}_{j=0}^{\infty}$ such that $\nu(\lambda_{\kappa_l}) = E_l$.

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- Define the index κ_I to be the largest index of $\{\lambda_j\}_{j=0}^{\infty}$ such that $\nu(\lambda_{\kappa_I}) = E_I$.
- Suppose we have a cochain $b_{(I)} \in C$ that solves the **Maurer-Cartan equation modulo terms in $F^{E_I} C$**, i.e.

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- Define the obstruction classes $o_j \in A^*(L)$ for $j = \kappa_l + 1 \ldots, \kappa_{l+1}$ to be,

$$o_j := \text{coefficient of } \lambda_j \text{ in } \mathfrak{m}^{\gamma}(e^{b_{(l)}})$$

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$$o_j := \text{coefficient of } \lambda_j \text{ in } \mathfrak{m}^{\gamma}(e^{b_{(l)}})$$

- The o_j are closed and satisfy $\deg o_j = 2 - \deg \lambda_j$.

Proof of classification of bounding pairs (for cohomology spheres), cont'd

- We prove the following proposition, which shows ρ is **surjective**: assuming $H^*(L;\mathbb{R}) \cong H^*(S^n;\mathbb{R})$, then for any closed $\gamma \in (\mathcal{I}_Q D)_2$ and any $a \in (\mathcal{I}_R)_{1-n}$, there exists a bounding cochain b for $\mathfrak{m}^\gamma$ such that $\int_L b = a$.

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- Idea of proof: the assumption that L is a **cohomology sphere** ensures the obstruction classes exact. We can then inductively build a bounding cochain that satisfies the Maurer-Cartan equation modulo $F^{E_l}C$. Taking the limit as $l \to \infty$, we get an honest bounding cochain satisfying the Maurer-Cartan equation.

Proof of classification of bounding pairs (for cohomology spheres), cont'd

- Proof: For the base case, take a representative of the Poincaré dual of a point $b_0 \in A^n(L)$. Set $b_{(0)} := a \cdot b_0 \in \mathcal{I}_R C$. By the energy zero property, $\mathfrak{m}^\gamma(e^{b_{(0)}}) \equiv 0 = c_{(0)} \cdot 1 \pmod{F^{E_0} C}$ where $c_{(0)} := 0$. Clearly, $\int_L b_{(0)} = a, db_{(0)} = 0$.

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- By induction, suppose we have $b_{(l)} \in C$ with $\deg_C b_{(l)} = 1$, and

$$\int_L b_{(l)} = a, \quad \mathfrak{m}^\gamma(e^{b_{(l)}}) \equiv c_{(l)} \cdot 1 \pmod{F^{E_0} C}$$

Proof of classification of bounding pairs (for cohomology spheres), cont'd

- Proof: For the base case, take a representative of the Poincaré dual of a point $b_0 \in A^n(L)$. Set $b_{(0)} := a \cdot b_0 \in \mathcal{I}_R C$. By the energy zero property, $\mathfrak{m}^\gamma(e^{b_{(0)}}) \equiv 0 = c_{(0)} \cdot 1 \pmod{F^{E_0} C}$ where $c_{(0)} := 0$. Clearly, $\int_L b_{(0)} = a, db_{(0)} = 0$.
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$$\int_L b_{(l)} = a, \quad \mathfrak{m}^\gamma(e^{b_{(l)}}) \equiv c_{(l)} \cdot 1 \pmod{F^{E_0} C}$$

- Take the obstruction chains o_j of $b_{(l)}$. We can choose forms $b_j \in A^{1-\deg \lambda_j}(L)$ such that $(-1)^{\deg \lambda_j} db_j = -o_j$ for all $j \in \{\kappa_l + 1, \ldots, \kappa_{l+1}\}$ with $\deg \lambda_j \neq 2$. If $\deg \lambda_j = 2 - n$, then $o_j = 0$ since $\deg o_j = 2 - \deg \lambda_j$. Hence we can take $b_j = 0$. If $2 - n < \deg \lambda_j < 2$, then $0 < |o_j| < n$, so the assumption that L is a **cohomology sphere** shows existence of the b_j. For other possible values of $\deg \lambda_j$, $o_j = 0$ by degree considerations, so we can again take $b_j = 0$.

Proof of classification of bounding pairs (for cohomology spheres), cont'd

- The **energy zero property** gives us,

$$b_{(l+1)} := b_{(l)} + \sum_{\kappa_l + 1 \leq j \leq \kappa_{l+1}, \deg \lambda_j \neq 2} \lambda_j b_j$$

which satisfies $\mathfrak{m}^{\gamma}(e^{b_{(l+1)}}) \equiv c_{(l+1)} \cdot 1 \pmod{F^{E_{l+1}C}}$ and $\int_L b_{l+1} = a$.

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- We get a sequence $\{b_{(l)}\}_{l=0}^\infty$ that converges in the filtration topology. Thus, $b := \lim_l b_{(l)}$ is our desired bounding cochain with $\mathfrak{m}^\gamma(e^b) = c \cdot 1$ and $\int_L b = a$.

Proof of classification of bounding pairs (for cohomology spheres), cont'd

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- Injectivity of ρ relies on a similar obstruction class argument.

Proof of the OGW axioms

- It is enough to prove the axioms for the basis elements as input. Without loss of generality, take $\Gamma_0 = 1, \Gamma_1, \ldots, \Gamma_N \in H_2(M, L; \mathbb{R})$ as a basis.
- (**Proof of degree axiom**): The superpotential $\Omega(\gamma, b)$ is of degree $3 - n$. The partial derivatives $\partial_{t_{i_1}} \ldots \partial_{t_{i_l}} \partial_s^k$ decrease the degree by $k \deg s + \sum_{j=1}^{l} 2 - |\Gamma_j|$. Taking out T^β decreases the degree by $\mu(\beta)$. When setting the variables $s = t_j = 0$, only the degree zero term remains. Thus $OGW_{\beta,k} \neq 0$ only if $(3 - n) - k(1 - n) - (\sum_{j=1}^{l} 2 - |\Gamma_j|) - \mu(\beta) = 0$.

Proofs of the OGW axioms, cont'd

- (**Proof of fundamental class axiom**): (We can assume $\partial_{t_0} b = 0$) We have

$$(-1)^n \partial_{t_0}\Omega = \sum_{k,l\geq 0} \frac{1}{l!(k+1)} \langle \partial_{t_0} \mathfrak{q}_{k,l}(b^{\otimes k}; \gamma^{\otimes l}), b\rangle + \partial_{t_0} \mathfrak{m}^{\gamma}_{-1}$$
$$= \sum_{k,l\geq 0} \frac{1}{(l-1)!(k+1)} \langle \mathfrak{q}_{k,l}(b^k; 1 \otimes \gamma^{l-1}), b\rangle + 0$$
$$= \langle \mathfrak{q}_{0,1}(;1), b\rangle$$
$$= (-1)^{n+1} T^{\beta_0} \int_L b := (-1)^{n+1} T^{\beta_0} s$$

- Thus, $\partial_J \partial_{t_0}\Omega|_{s=t_j=0} \neq 0$ unless $J = \{s\}$, in which case it is $-T^{\beta_0}$. By definition, this means $OGW_{\beta_0,1}(1) = -1$, and 0 otherwise.

Proof of the OGW axioms, continued

- (**Proof of symplectic deformation invariance**): Define Λ^J to be the J-dependent Novikov ring,

$$\Lambda^J := \{\sum_{i=0}^{\infty} a_i T^{\beta_i} \in \Lambda | \forall i, \exists \text{ J-holomorphic disc representing } \beta_i\}$$

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- Take a neighborhood U of ω in which J is ω'-tame for all $\omega' \in U$. We can similarly define A^∞ operations $\mathfrak{m}_\gamma^J$ that use the J-dependent Novikov ring. Furthermore, we can find a bounding pair (γ, b) such that b is a bounding cochain for $\mathfrak{m}_\gamma^J$, and $([\gamma], \int_L b) = (\Gamma, s)$.

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- Take a neighborhood U of ω in which J is ω'-tame for all $\omega' \in U$. We can similarly define A^∞ operations $\mathfrak{m}^J_\gamma$ that use the J-dependent Novikov ring. Furthermore, we can find a bounding pair (γ, b) such that b is a bounding cochain for $\mathfrak{m}^J_\gamma$, and $([\gamma], \int_L b) = (\Gamma, s)$.
- The bounding cochain b depends on ω' only through Λ^J, which is the same for all $\omega' \in U$ and J. Hence b is a bounding cochain for $\mathfrak{m}^J_\gamma$ for all $\omega' \in U$.

Proof of the OGW axioms, continued

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- The bounding cochain b depends on ω' only through Λ^J, which is the same for all $\omega' \in U$ and J. Hence b is a bounding cochain for $\mathfrak{m}^J_\gamma$ for all $\omega' \in U$.
- But b is a bounding cochain for the $\{\mathfrak{m}^\gamma_k\}$ in defining OGWs for all $\omega' \in U$, since $\mathfrak{m}^\gamma_k$ only considers classes that can be represented by J-holomorphic dics. But this implies the superpotential $\Omega(\gamma, b)$ and hence $OGW_{\beta,k}$ is constant for all $\omega' \in U$.

Final Remarks

- S-T showed that when there's an anti-symplectic involution, their definition of OGWs generalize Welshinger's and Georgieva's invariants.
- In a subsequent paper, S-T show their superpotential satisfy open WDVV equations.

Thanks!