Spacetime Geometries

by

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1 Introduction

This paper is my senior honors thesis, written under the direction of Professor András Vasy. The honors in mathematics is designed to allow a student to explore and provide an exposition of an advanced topic. I chose to study the Einstein field equations in the theory of general relativity. General relativity is a beautiful subject, and I only regret that I did not take a course of it in my time at Stanford. However, under the guidance of Professor Vasy, I was able to learn a substantial amount.

The theory of general relativity revolutionized the way we think about gravity. Einstein's theory gave us the mathematical tools to explain and predict physical phenomena in the theory of black holes and cosmology. It combines space and time into a single entity, represented by a spacetime manifold, and introduces a curved Lorentzian metric on it to represent the effects of gravity imposed by matter energy. The aim of this thesis is to provide an exposition of several spacetimes obtained by the Einstein field equations in general relativity, which are a set of physically meaningful equations that mathematically represent the relationship between gravity and matter energy. The field equations are presented as,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{1}$$

where R_{ab} is called the Ricci tensor, R the scalar curvature, Λ the cosmological constant, g_{ab} the spacetime metric, and T_{ab} the energy momentum tensor. We shall provide some key preliminaries in Riemannian and Lorentzian geometry to explain these mathematical objects. With sufficient background, we shall then introduce the Einstein field equations by deriving them through variational methods. When solving the field equations, one seeks to ultimately solve for the metric that the underlying manifold is endowed with. Such solutions are called exact solutions, and we will explore several different ones, each obtained under imposed assumptions. We will examine what happens to geodesics in these spacetimes by making necessary coordinate transformations into conformal spacetimes. Conformal transformations preserve the behavior of null geodesics, and allow us to study the structure of infinity and existence of singularities of the spacetime manifold more carefully.

Throughout this thesis, we shall adopt the Einstein summation convention, which is a concise way to denote summation over a particular index that occurs as both a superscript and a subscript. For example, suppose we have a tensor of order two h_{ik} given by $h_{ik} = \sum_j h_i^j g_{jk}$. The Einstein summation convention would allow us to not write the summation symbol and just denote,

$$h_{ik} = h_i^j g_{jk} \tag{2}$$

The notation in general relativity can sometimes involve many indices, and this convention provides a clearer way to write expressions.

The images in this thesis were all acquired in Hawking's The Large Scale Structure of Space-Time and cited accordingly.

2 Mathematical Preliminaries

In this section, we discuss some necessary prerequisites in Riemannian and Lorentzian geometry. The definitions and results henceforth were acquired through Hawking's The Large Scale Structure of Space-time and Kuhnel's Differential Geometry books.

2.1 Manifolds

Recall that an n-dimensional, C^r manifold M is a topological space that is locally similar to $\mathbb{R}^n$ at each point. There exists an atlas or a collection of charts (U_α, ϕ_α) where the U_α are open subsets of M and ϕ_α is a homeomorphic map from U_α to open subsets of $\mathbb{R}^n$. Furthermore, $\bigcup_\alpha U_\alpha = M$, and if $U_\alpha \cap U_\beta = \emptyset$ for some α, β, then $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a C^r map from an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$. Examples of manifolds include any open subset U of $\mathbb{R}^n$, the torus $\mathbb{R}^2/\mathbb{Z}^2$, or the real projective plane $\mathbb{RP}^2$. M is compact if for every cover of the manifold, there exists a finite sub-cover. It is orientable if there is an atlas (U_α, ϕ_α) such that in every non-empty intersection $U_\alpha \cap U_\beta$, the determinant of the Jacobian $\{\partial x^i / \partial x^{'j}\}$ is positive, where $\{x^n\}$ and $\{x^{'n}\}$ are local coordinates of U_α and U_β respectively. It is Hausdorff if for every pair of distinct points $p, q \in M$, there exist neighborhoods U, V of p, q respectively such that $U \cap V = \emptyset$. Given an atlas $\{U_\alpha, \phi_\alpha\}$, a partition of unity is a set of functions $\{g_\alpha\}$ such that $0 \leq g_\alpha \leq 1$, the support of g_α is contained in the corresponding U_α, and $\sum_\alpha g_\alpha(p) = 1$ for all $p \in M$. A C^r extension of M is another manifold M' and an isometric C^r embedding $\mu : M \to M'$ i.e. μ is a homeomorphism of M onto its image $\mu(M)$. If there is no such extension, then we say that M is inextendible.

2.2 Vectors, forms, and tensors

With a manifold, we can impose notions of vector fields and tensor fields on it. However, since we can no longer assume there is an ambient space (such as $\mathbb{R}^n$) for general manifolds, our definitions must be constructed intrinsically. First, a C^k curve $\lambda(t)$ on M is simply a C^k map from the real line $\mathbb{R}^1$ into M. Given a function f on M i.e. a map from M to $\mathbb{R}^1$, we define the vector $(\partial/\partial t)_\lambda|_{t_0}$ tangent to a C^1 curve $\lambda(t)$ on M as an operator that maps f at $\lambda(t_0)$ to the derivative of f in the direction $\lambda(t)$. More explicitly, applying the operator to f,

$$(\frac{\partial f}{\partial t})_\lambda|_t = \lim_{s \to 0} \frac{1}{s}(f(\lambda(t+s)) - f(\lambda(t)) \tag{3}$$

This is similar to the usual notion of a derivative.

If we have local coordinates $\{x^n\}$, which are guaranteed by the charts in a neighborhood of some $p \in M$, then the coordinate derivatives,

$$(\partial/\partial x^1)|_p, ..., (\partial/\partial x^n)|_p \tag{4}$$

form a basis for the tangent space T_pM, or the set of all tangent vectors at p. Furthermore, the tangent space is a vector space of dimension n. It is now easy to define a vector field on the manifold M as a mapping $X : p \in M \to X_p \in T_pM$ from each point of M to its tangent space with the requirement that if $\{x^n\}$ are local coordinates around p, then the coefficients $\psi^i(p)$ in the representation $X_p = \sum_{i=1}^n \psi^i(p)\frac{\partial}{\partial x^i}$ are differentiable, real valued functions.

With vectors in hand, the next entity we discuss are forms. A one-form ω is a real valued linear functional on the space T_pM for $p \in M$. For some $X \in T_pM$, we write $\langle \omega, X\rangle$ as the number in which ω maps X to. Linearity implies that for $X, Y \in T_pM$ and $\alpha, \beta \in \mathbb{R}$,

$$\langle \omega, \alpha X + \beta Y\rangle = \alpha\langle \omega, X\rangle + \beta\langle \omega, Y\rangle \tag{5}$$

Given a basis $\{E_a\}$ of vectors at p, we can define a unique set of one-forms $\{E^a\}$, where E^i maps any vector $X \in T_pM$ to its i^{th} component X^i, as $X = \sum_{i=1}^n X^i E_i$. This set of one-forms in fact forms a basis for any one-form ω, hence $\omega = \sum_{i=1}^n \omega_i E^i$, and is called the dual basis to $\{E_a\}$. The set of one-forms spanned by the dual basis is called the dual space T_p^* of the tangent space T_p. Using this decomposition, we have,

$$\langle \omega, X\rangle = \langle \sum_{i=1}^n \omega_i E^i, \sum_{j=1}^n X^j E_j\rangle = \sum_{k=1}^n \omega_k X^k \tag{6}$$

Each function f on M defines a one form df at p by the rule $\langle df, X\rangle = Xf$, and is called the differential of f. If we have local coordinates $\{x^n\}$, then the set of differentials $(dx^1, dx^2, ..., dx^n)$ will be the dual basis to the coordinate derivatives $(\partial/\partial x^1, \partial/\partial x^2, ..., \partial/\partial x^n)$, and satisfies the relation $\langle dx^i, \partial/\partial x^j\rangle = \partial x^i/\partial x^j = \delta^i_j$

We now turn to tensors, which give us a compact way of theoretically expressing many ideas in mathematical physics. We construct a tensor of type (r, s) as a real valued function on the following Cartesian product: at $p \in M$, we form $\Pi^s_r = T_p^* \times T_p^* ... \times T_p^* \times T_p \times T_p ... \times T_p$, where there are r copies of T_p^* and s copies of T_p. In each argument, the tensor is linear. The space of all tensors at some p is called the tensor product,

$$T^r_s(p) = T_p \otimes ... \otimes T_p \otimes T_p^* \otimes ... \otimes T_p^* \tag{7}$$

where there are r copies of T_p and s copies of T_p^*; in particular $T^1_0(p) = T_p$ and $T^0_1(p) = T_p^*$. We can add and multiply tensors in the obvious way, which makes the tensor product into a real vector space of dimension n^{r+s}.

We can also perform the outer product on tensors, namely given tensors $R \in T^r_s(p)$ and $S \in T^p_q(p)$, the tensor $R \otimes S \in T^{r+p}_{s+q}(p)$ is the element which maps $(\eta^1, ..., \eta^{r+p}, Y_1, ..., Y_{s+q}) \in \Pi^{s+q}_{r+p}$ to the number,

$$R(\eta^1, ..., \eta^s, Y_1, ..., Y_r)S(\eta^{s+1}, ..., \eta^{s+q}, Y_{r+1}, ..., Y_{r+p}) \tag{8}$$

With the outer product, one can obtain a basis for the tensor product $T^r_s(p)$; namely, given dual bases $\{E_a\}$ and $\{E^a\}$ of T_p and T_p^*, the set $\{E_{a_1} \otimes ... \otimes$

$E_{a_r} \otimes E^{b_1} \otimes ... \otimes E^{b_s}\}$ where a_i and b_j run from 1 to n, is a basis for the tensor product. Thus, any tensor $T \in T^r_s(p)$ can expressed in terms of this basis as,

$$T = T^{a_1...a_r}_{b_1...b_s} E_{a_1} \otimes ... \otimes E_{a_r} \otimes E^{b_1} \otimes ... \otimes E^{b_s} \tag{9}$$

where $T^{a_1...a_r}_{b_1...b_s}$ are called the components of the tensor T and are given by $T^{a_1...a_r}_{b_1...b_s} = T(E^{a_1}, ..., E^{a_r}, E_{b_1}, ..., E_{b_s})$.

The contraction of a type (r, s) tensor T with components $T^{a_1...a_r}_{b_1...b_s}$ with respect to the bases $\{E_a\}$ and $\{E^a\}$ on the first contravariant and covariant indices is defined to be the tensor C^1_1 of type $(r-1, s-1)$ given by,

$$C^1_1 = T^{a_1 a_2 ... a_r}_{a_1 b_2 ... b_s} E_b \otimes ... \otimes E_d \otimes E^f \otimes ... \otimes E^g \tag{10}$$

This operation will prove useful in quantities relating to the curvature of manifolds in the field equations.

We can also define the symmetric and antisymmetric part of a tensor on a given set of covariant and contravariant indices. The symmetric part of a tensor is that tensor whose components are,

$$T^{b...f}_{(a_1...a_r)} = \frac{1}{r!} H \tag{11}$$

where H denotes the set {sum over all permutations of the indices a_1 to a_r $(T^{b...f}_{a_1...a_r})$}. Similarly, we denote the antisymmetric part of a tensor as that tensor whose components are,

$$T^{b...f}_{[a_1...a_r]} = \frac{1}{r!} J \tag{12}$$

where J denotes the set {alternating sum over all permutations of the indices a_1 to a_r $(T^{b...f}_{a_1...a_r})$}.

It is now easy to describe what a tensor field is. Like a vector field, a tensor field of type (r, s) is an assignment $X : p \in M \to X_p \in T^r_s(p)$ of each point to a tensor X_p, where we require that the components of X_p with respect to any basis are differentiable functions.

2.3 Differentiation and integration on manifolds

We shall briefly review some ideas of extending calculus from $\mathbb{R}^n$ onto more general manifolds. Since the Einstein field equations involve a set of nonlinear partial differential equations on an abstract manifold M, we are interested in differentiation on M. Recall that an p-form is simply a $(0, p)$ tensor that is antisymmetric on all p indices. If A and B are p- and q-forms respectively, we can define a $(p+q)$-form $A \wedge B$ from them, where $\wedge$ is the skew symmetrized tensor product $\otimes$; that is, $A \wedge B$ is that tensor of type $(0, p+q)$ with components determined by,

$$(A \wedge B)_{a...bc...f} = A_{[a...b} B_{c...f]} \tag{13}$$

One approach to define differentiation is to use the exterior differentiation operator d on p-form fields. Acting on a zero-form field (i.e. a function f), it gives the one form field df defined by,

$$\langle df, X\rangle = Xf \tag{14}$$

for all vector fields X. Acting on a r-form field,

$$A = A_{12...r}dx^1 \wedge dx^2 \wedge ... \wedge dx^r \tag{15}$$

it gives the $(r+1)$ form field dA defined by,

$$dA = dA_{12...r} \wedge dx^1 \wedge dx^2 \wedge ... \wedge dx^r \tag{16}$$

We can naturally define integration of n-forms over the manifold. Assume that the manifold is compact, orientable, and has dimension n. We use the atlas (U_α, ϕ_α) of the manifold to map each open set U_α into $\mathbb{R}^n$ and then perform a Lebesgue or Riemannian integral there, i.e. if A is an p-form field on M, then the integral of A over M is defined as,

$$\int_M A = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} f_\alpha A_{12...p} dx^1 dx^2 ... dx^p \tag{17}$$

where $\{f_\alpha\}$ is a partition of unity of the atlas, and $A_{12...p}$ are the components of A. With the exterior derivative, we can generalize Stokes' Theorem which states that if B is an $(n-1)$-form field on M, then,

$$\int_{\partial M} B = \int_M dB \tag{18}$$

which in its proof is essentially a more general form of the fundamental theorem of calculus.

The second approach in defining differentiation is through the Lie derivative. Motivated by the fundamental theorem of systems of ordinary differential equations, if X is a vector field on M, then there is an unique curve $\lambda(t)$ through each point $p \in M$ such that $\lambda(0) = p$ and $\lambda(t) = X|_{\lambda(t)}$. If $\{x^i\}$ are local coordinates so that the curve $\lambda(t)$ has coordinates $x^i(t)$ and the vector X has components X^i, this curve is known as an integral curve and is the solution of the system of differential equations,

$$\partial x^i/\partial t = X^i(x^1(t), ..., x^n(t)) \tag{19}$$

Thus, for each point $q \in M$, there is an open neighborhood U of q and an $\epsilon > 0$ such that the vector field X defines a family of diffeomorphisms $\phi_t : U \to M$ obtained by taking each point $p \in U$ a parameter distance t along the integral curves of X. In fact, the ϕ_t is actually a group under composition of diffeomorphism; hence $\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t$, $\phi_{-t} = (\phi_t)^{-1}$, and ϕ_0 is the identity element. We define the Lie derivative $L_X T$ of a tensor field T with respect to X as,

$$L_X T|_p = \lim_{t\to 0} \frac{T|_p - \phi_{t^*} T|_p}{t} \tag{20}$$

2.4 The Metric

The metric is one of the key objects of study in general relativity, and is the tool that allows one to measure lengths, angles, and study the causal structure of spacetime manifolds. It is a symmetric tensor field of type $(0,2)$ denoted by g_{ab}. At each point p in the manifold M, the metric takes as input two vectors in T_p, and returns a scalar value. We can associate a magnitude of a vector $X \in T_p$ by computing the value $|g(X,X)|^{\frac{1}{2}}$. The angle between two vectors $X, Y \in T_p$ is measured by,

$$\frac{g(X,Y)}{|g(X,X)g(Y,Y)|^{\frac{1}{2}}} \tag{21}$$

Given a piecewise C^1 curve $\gamma(t)$ connecting two points a and b with tangent vector $\frac{\partial}{\partial t}$ such that $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ has the same sign at all points along $\gamma(t)$, then the path length is defined to be,

$$L = \int_a^b |g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})|^{\frac{1}{2}} dt \tag{22}$$

The components of the metric can be evaluated since the metric is itself a tensor; with respect to a basis $\{E_a\}$, the components are,

$$g_{ab} = g(E_a, E_b) = g(E_b, E_a) \tag{23}$$

We assume the metric to be non-degenerate at each point $p \in M$, which means that there is no non-zero vector $X \in T_p$ such that $g(X,Y) = 0$ for all vectors $Y \in T_p$. We can then define a unique symmetric tensor of type $(2,0)$ denoted by g^{ab}, which we shall call the inverse metric tensor or just the inverse metric. The inverse metric has the property that its components with respect to the basis $\{E_a\}$ dual to the basis $\{E^a\}$,

$$g^{ab} g_{bc} = \delta^a_c \tag{24}$$

The metric tensor can be thought of as a matrix since it is of order two. Translating to the language of linear algebra, the assumption of non-degeneracy gives the non-singular properties of the matrices associated to the inverse/metric tensors. Thus, the metric g_{ab} and inverse metric g^{ab} can be used to give an isomorphism between any covariant tensor argument and any contravariant argument, or to "raise/lower indices". For example, given a contravariant vector X^a, the uniquely associated covariant vector is $X_a = g_{ab} X^b$.

There is a way to classify the metric g_{ab} by looking at the eigenvalues of its corresponding matrix. The signature of g_{ab} is defined to be the number of positive eigenvalues minus the number of negative ones. If g_{ab} is non-degenerate and continuous, then the signature will be constant on all of M; by a suitable choice of a basis $\{E_a\}$, the metric components can at any point p be brought to the form,

$$g_{ab} = diag(+1, +1, ..., +1, -1, ..., -1) \tag{25}$$

where there are $\frac{1}{2}(n+s)$ positive terms and $\frac{1}{2}(n-s)$ negative terms, n is the dimension of the manifold, and s is the signature of the metric. When the metric is of signature n, it is called positive definite, which means that $g(X,X)=0 \Rightarrow X=0$. When the metric is of signature $(n-2)$, the metric is called Lorentz.

2.5 Causality

With the Lorentz metric, we can in fact classify tangent vectors at $p \in M$ into three categories. Given a vector $X \in T_p$, we say it is timelike, null, or spacelike according to whether $g(X,X)$ is negative, zero, or positive. Since the metric is smooth, there is a boundary between the set of spacelike vectors and the set of timelike vectors. This boundary will form a double cone and is called the light cone (the reason of this naming is due to the well known postulate in relativity stating that information cannot travel faster than light) or the set of vectors X where $g(X,X)=0$. Naturally, a curve is called timelike, null, or spacelike, if every tangent vector on the curve is respectively such.

The spacetime manifold is called time-orientable if there exists a continuous division of non-spacelike vectors into two classes, which we label as future- and past-directed. If the manifold is time-orientable, then we can label non-spacelike curves as either future- or past-directed if every tangent vector of the curve is future- or past-directed.

Given an open set S in our spacetime, we define the future Cauchy development of S as the set of points $p \in M$ such that every past directed inextendible non-spacelike curve through p intersects S. This set is denoted by $D^+(S)$, and consequently it is easy to see that $D^+(S) \subset S$. The past Cauchy development is defined in a similar way, and is denoted by $D^-(S)$. These sets are of importance, since physically these regions represent the domains of influence or dependence of S. Events can only be causally related if they can be joined by a non-spacelike curve.

S is called a Cauchy surface if the set of points $D^+(S) \cup D^-(S) = M$. Thus, every non-spacelike curve in the spacetime intersects the surface S. We shall see that the existence of a Cauchy surface is a property of the spacetime, since not all spacetimes have Cauchy surfaces. When a spacetime does admit one, it is called globally hyperbolic. Cauchy surfaces are of interest since they allow one to predict the state of the spacetime at any time in the past or future given initial data. However, realistically one can only predict to the future, and physically there may be extra information coming into the domain of interest that will upset the initial data. But these surfaces are still of tremendous theoretical interest.

2.6 Connections and the Covariant Derivative

We want to generalize the derivative in order to operate on abstract manifolds and set up the Einstein field equations. The partial derivative is too limited in the sense that it depends on a direction at the point of interest. Similarly, the Lie derivative depends on the direction of the vector field X at some point as well as at neighboring points. By using connections, we introduce an extra structure onto the manifold that allows us to achieve the desired generality.

A connection ∇ is an operator which maps two vector fields X and Y to a third vector field $\nabla_X Y$ such that the following conditions are satisfied ($f : M \to R$ denotes a real differentiable function),

$$\begin{aligned}
\nabla_{X_1+X_2} Y &= \nabla_{X_1} Y + \nabla_{X_2} Y \\
\nabla_{fX} Y &= f \nabla_X Y \\
\nabla_X (Y_1 + Y_2) &= \nabla_X Y_1 + \nabla_X Y_2 \\
\nabla_X (fY) &= f \nabla_X Y + X(f) Y
\end{aligned}$$

where $X(f)$ refers to the directional derivative of f in the direction X. A connection that is of particular interest to us in general relativity is the Levi-Civita connection. In addition to the properties above, a Levi-Civita connection also satisfies the following conditions called metric compatibility and torsion-freeness, respectively,

$$\begin{aligned}
\nabla_X (g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\
[X, Y] &= \nabla_X Y - \nabla_Y X
\end{aligned}$$

where $[X, Y] \triangleq L_X Y$. One of the remarkable things about Levi-Civita connections is that there always exists a unique such connection on a manifold M with metric g.

The primary reason for introducing connections was to able to take more generalized derivatives of tensors. There are certain properties that we want to endow the derivative through construction. First, we would like the derivative (which we shall denote by ∇) to satisfy familiar linearity and product rules when applied to two tensors T and S,

$$\begin{aligned}
\nabla(T + S) &= \nabla T + \nabla S \\
\nabla(T \otimes S) &= \nabla T \otimes S + T \otimes \nabla S
\end{aligned}$$

Second, it should simply reduce to the usual partial derivative when applied to scalar functions i.e. for scalar functions f,

$$\nabla_\mu f = \partial_\mu f \tag{26}$$

where ∇ is taken with respect to some index μ. Third, we want the derivative to transform like a tensor.

We shall construct the covariant derivative by first considering what the derivative should be when taken with just a contravariant vector, and then just a covariant vector. A generalization to arbitrary tensors will then follow. The above considerations and desired properties prompt us to define the covariant derivative of a contravariant vector as,

$$\nabla_\mu V^\nu \triangleq \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{27}$$

where the $\Gamma^{\eta}_{\mu\lambda}$ are called the connection coefficients or Christoffel symbols. Equation (27) is essentially saying that the covariant derivative of a vector is the linear combination of the usual partial derivative of the vector and some correction term. We shall determine how to define the Christoffel symbols, which are $n \times n$ matrices, in order for the covariant derivative to transform like a tensor i.e. we want the following transformation law for covariant derivatives,

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \nabla_{\mu} V^{\nu} \tag{28}$$

To do this, let's expand both sides of (28). Expanding the left side first,

$$\begin{aligned}\nabla_{\mu'} V^{\nu'} &= \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} \\ &= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} V^{\lambda}\end{aligned}$$

The right hand side can be likewise expanded to get,

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \nabla_{\mu} V^{\nu} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} V^{\lambda} \tag{29}$$

Equating (28) and (29) and noticing that the above must hold for any vector V^{λ}, we arrive at the definition for the transformation law of Christoffel symbols in order for the covariant derivative to transform like a tensor,

$$\Gamma^{\nu'}_{\mu'\lambda'} \triangleq \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}} \tag{30}$$

Now, given the covariant derivative of a contravariant vector V^{λ}, the covariant derivative of a covariant vector ω_{μ} will be given through some algebraic manipulations as,

$$\nabla_{\mu} \omega_{\nu} = \partial_{\mu} \omega_{\nu} - \Gamma^{\lambda}_{\mu\nu} \omega_{\lambda} \tag{31}$$

Thus, the covariant derivative of an arbitrary tensor is defined as follows: for each upper index we introduce a + connection coefficient, and for each lower index we introduce a - connection coefficient,

$$\begin{aligned}\nabla_{\sigma} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} &= \partial_{\sigma} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} \\ &+ \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1\lambda\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} + \ldots - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}_{\lambda\nu_2\ldots\nu_l} \\ &- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\lambda\ldots\nu_l} - \ldots - \Gamma^{\lambda}_{\sigma\nu_l} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\lambda}\end{aligned}$$

Notationally, we denote the covariant derivative of an arbitrary tensor as,

$$\nabla_{\sigma} T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} \equiv T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l;\sigma} \tag{32}$$

From the formula above, we see that the connection coefficients completely determine the derivatives of arbitrary tensors. Hence, if we understand the connection coefficients, we can understand derivatives. In particular, as general relativity is mainly concerned with Levi-Civita connections and there exists a unique Levi-Civita on a manifold (M, g), when one uses the coordinate basis $\{\frac{\partial}{\partial x^\mu}\}$ to express tensors, the Christoffel symbols can be written solely in terms of first derivatives of the metric,

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{33}$$

As the Levi-Civita connection is torsion free, the Christoffel symbols have the symmetry,

$$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} \tag{34}$$

With the Christoffel symbols, we can now define several more important tensors in Riemannian geometry and general relativity. The Riemann curvature tensor gives a measure of non-commutation of covariant derivatives. Given vector fields X, Y, Z, the curvature tensor $R(X, Y)Z$ is defined by,

$$R(X, Y)Z \triangleq \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \tag{35}$$

When expressed in the coordinate basis, it can be written in terms of the Christoffel symbols,

$$R^a_{bcd} = \frac{\partial \Gamma^a_{db}}{\partial x^c} - \frac{\partial \Gamma^a_{cb}}{\partial x^d} + \Gamma^a_{cf}\Gamma^f_{db} - \Gamma^a_{df}\Gamma^f_{cb} \tag{36}$$

The curvature tensor has several symmetries, namely,

$$\begin{aligned}
R^a_{bcd} &= -R^a_{bdc} \\
R^a_{bcd} &= -R^b_{acd} \\
R^a_{bcd} &= R^c_{dab} \\
R^a_{bcd} + R^a_{dbc} + R^a_{cdb} &= 0 \\
R^a_{bcd;e} + R^a_{bec;d} + R^a_{bde;c} &= 0
\end{aligned}$$

The last symmetry above is known as the Bianchi identity.

By contracting the curvature tensor, one obtains the Ricci tensor of type $(0, 2)$ with components,

$$R_{bd} = R^a_{bad} \tag{37}$$

Since the Riemann curvature tensor is symmetric in the pairs of indices $\{ab\}$,$\{cd\}$, the Ricci tensor is also symmetric in its indices,

$$R_{ab} = R_{ba} \tag{38}$$

The curvature scalar R is given by the contraction of the Ricci tensor,

$$R = R^a_a = g^{bd} R_{bd} \tag{39}$$

2.7 Parallel Transport and Geodesics

Parallel Transport is an idea that seeks to measure the curvature of the manifold by measuring how much tensors "change" when moved along curves. More precisely, let T be a C^r tensor defined along a curve $x^\mu(\lambda)$. We define another operator related to the covariant derivative,

$$\frac{\partial D}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu \tag{40}$$

The parallel transport of a tensor T along $x^\mu(\lambda)$ is defined to be,

$$(\frac{D}{d\lambda}T)^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} \equiv \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}_{\nu_1\nu_2\ldots\nu_l} = 0 \tag{41}$$

For a vector V^μ, the equation of parallel transport takes the form,

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0 \tag{42}$$

The parallel transport equation is a first order differential equation with initial data: given a tensor T along the curve, there will be a unique continuation of T to other points on the curve such that the continuation still solves (34).

With parallel transport, we can now discuss geodesics, which are the curved space generalizations of "straight lines". Intuitively, geodesics should be the paths of least distance, or paths that parallel transport their own tangent vector. Thus, if $x^\mu(\lambda)$ is again a curve, and $\frac{dx^\mu}{d\lambda}$ its tangent vector, then the equation for parallel transport given by above is,

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0$$
$$\Rightarrow \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

Standard theorems from ordinary differential equations state that for any point $p \in M$ and any vector X_p at p, there exists a unique geodesic $\lambda_{X_p}(t)$ with starting point p and initial direction X_p. Thus, we can define the exponential map $\exp : T_p \to M$ which takes as input $X \in T_p$ and returns the point on the manifold a unit parameter distance along $\lambda_{X_p}(t)$. It is important to notice that the exponential map may not be defined on all of T_p, since the unique geodesic may run into singularities of the manifold for some tangent vectors. However, if the parameter t takes on all values i.e the geodesic $\lambda_{X_p}(t)$ does not run into any singularities, the geodesic is called complete. The manifold M is called geodesically complete if all geodesics on M are complete i.e. if the exponential map is defined on all of T_p for every point p of M.

3 General Relativity and the Einstein field equations

One of the key insights Einstein made to create General Relativity was not to consider space and time as separate entities, but to unite them as simply spacetime. This insight provided the mathematical foundation to formalize amazing and beautiful physical phenomenon such as gravitational lensing and black hole formation. We consider spacetime as a manifold (M, g) that is connected, Hausdorff, and C^∞. It is taken to be connected, since we would have no knowledge of disconnected regions of the universe. It is taken to be Hausdorff as this seems to accord with normal experience. We endow the spacetime with a Lorentz metric g. We also assume our manifold is inextendible; the reason for this is that we want to include all non-singular points in our space-time. If our space-time were extendible, then those points would just be regarded in our universe as well. Under these mathematical considerations, we shall use tools from Riemannian and Lorentzian geometry to analyze some amazing properties of spacetime.

Physically, there might be matter energy content in spacetime represented by various fields on our manifold. These fields can stem from electromagnetic, scalar, or perfect fluid sources. To represent this, we say that there exists a symmetric, order two tensor T^{ab} called the energy-momentum or stress energy tensor that encapsulates the matter energy content to be used in the Einstein field equations. Furthermore, given the Lagrangian of energy sources, it is possible to use variational methods to derive the associated energy-momentum tensor. For example, if the energy content is that of an electromagnetic field, then from electrodynamics the field can be described by a one-form A called the potential. The Lagrangian is thus given by,

$$L = \frac{-1}{8\pi} F_{ab} F_{cd} g^{ac} g^{bd} \tag{43}$$

where the tensor F is defined to be $2dA$, i.e. $F_{ab} = 2A_{[b;a]}$. From the Euler-Lagrange equations, the associated energy-momentum tensor is,

$$T_{ab} = \frac{1}{4\pi}(F_{ac} F_{bc} g^{cd} - \frac{1}{4} g_{ab} F_{ij} F_{kl} g^{ik} g^{jl}) \tag{44}$$

Before motivating the field equations, we first have to make some physical assumptions about the matter fields, since the theory we shall describe should agree with empirical evidence. To encode the physical law that information cannot travel faster than light, we require that a local causality postulate holds. Namely, if U is a convex neighborhood, then any points $p, q \in U$ can be joined by a curve $\lambda(t)$ such that it lies entirely in U and is non-spacelike. Pertaining to the matter fields, let $p \in U$ be some point in which every past directed non-spacelike curve through p intersects some surface $\Delta \subset U$. Let F be the set of points in Δ that are reached by such curves. Then, we require that the matter fields at p are uniquely determined by the values of the matter fields and their derivatives on F.

The second postulate we make is local conservation of energy and momentum, which agrees with physical predictions. Relating to the energy-momentum tensor T_{ab}, the covariant derivative of the tensor should be zero i.e. $T_{ab;b} = 0$. Also, we require that the tensor vanishes in an open set U if and only if all matter fields vanish in U.

The third postulate we make is that the Einstein field equations hold on the spacetime manifold we are considering. This will allow us to analyze the behavior of spacetimes under different initial conditions.

3.1 The field equations

Experimental evidence suggests that light is bent by large bodies, which is a phenomenon known as gravitational lensing. Therefore, there is empirical reason to believe that the field equations should relate the curvature of spacetime to the energy-momentum tensor. The equations are presented as,

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = 8\pi T_{ab} \tag{45}$$

where R_{ab} is the Ricci tensor, R is the curvature scalar, g_{ab} is the metric, T_{ab} is the energy momentum tensor, and Λ is the cosmological constant. We assume that the connection we are using to define the relevant tensors is a Levi-Civita connection, which we recall is metric compatible and torsion free. Since the tensors in the equation are symmetric and of order two, initially there seem to be ten second order nonlinear partial differential equations resulting from these considerations alone. However, using the fact that the covariant derivative of the energy-momentum tensor must be zero, we have,

$$(R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab})_{;b} = 0 \tag{46}$$

which reduces the number of equations we must solve down to six.

We want to solve the field equations under certain conditions and symmetries imposed on the spacetime to get the resulting metric g_{ab}. However, since two space-time metrics g_1 and g_2 are said to be equivalent if there exists a diffeomorphism taking g_1 to g_2, solving the field equations in fact gives us an equivalence class of metrics by diffeomorphism.

The field equations can be deduced as the Newtonian limit of a weak gravitational field by noting that general relativity must reduce to Poisson's equation in classical mechanics relating the gravitational potential to the mass density,

$$\Delta\phi = 4\pi G\rho \tag{47}$$

We take a variational approach to derive the field equations by looking at the Einstein-Hilbert action,

$$I = \int_D \sqrt{-|g|}(R - 2\Lambda)dv \tag{48}$$

where R is the scalar curvature, Λ is the cosmological constant, $|g|$ is the determinant of the metric, and the integral is taken over a four dimensional region D. We require that this action be stationary under variations i.e.

$$\delta I = \delta \int_D \sqrt{-|g|}(R - 2\Lambda)dv = 0 \tag{49}$$

We have that,

$$\begin{aligned}
\delta I &= \delta\left(\int_D \sqrt{-|g|}(R - 2\Lambda)dv\right) \\
&= \delta\left(\int_D \sqrt{-|g|}(g^{ab}R_{ab} - 2\Lambda)dv\right) \\
&= \int_D \sqrt{-|g|}g^{ab}(\delta R_{ab})dv + \int_D \sqrt{-|g|}R_{ab}(\delta g^{ab})dv + \int_D (R - 2\Lambda)(\delta\sqrt{-|g|})dv \\
&= \delta I_1 + \delta I_2 + \delta I_3
\end{aligned}$$

First, consider the variation of δI_1. By definition, the variation of the Ricci tensor is,

$$R_{ab} = R^c_{acb} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ac}$$

Hence,

$$\begin{aligned}
\delta R_{ab} &= \partial_c \delta\Gamma^c_{ab} - \partial_b \delta\Gamma^c_{ac} + \Gamma^d_{ba}\delta\Gamma^c_{cd} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^c_{bd}\delta\Gamma^d_{ac} \\
&= (\partial_c \delta\Gamma^c_{ab} + \Gamma^c_{cd}\delta\Gamma^d_{ba} - \Gamma^d_{ac}\delta\Gamma^c_{bd} - \Gamma^d_{bc}\delta\Gamma^c_{ad}) \\
&- (\partial_b \delta\Gamma^c_{ac} + \Gamma^c_{bd}\delta\Gamma^d_{ac} - \Gamma^d_{ba}\delta\Gamma^c_{cd} - \Gamma^d_{bc}\delta\Gamma^c_{ad})
\end{aligned}$$

Using the formula for covariant derivatives of tensors, the above is precisely,

$$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac} \tag{50}$$

δI_1 becomes,

$$\begin{aligned}
\delta I_1 &= \int_D \sqrt{-|g|}g^{ab}(\nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ac})dv \\
&= \int_D \sqrt{-|g|}[\nabla_c(g^{ab}\delta\Gamma^c_{ab}) - \delta\Gamma^c_{ab}\nabla_c g^{ab} - \nabla_b(g^{ab}\delta\Gamma^c_{ac}) + \delta\Gamma^c_{ac}\nabla_b g^{ab}]dv
\end{aligned}$$

Remembering that the covariant derivative of the metric is zero, we have,

$$\begin{aligned}
&= \int_D \sqrt{-|g|}[\nabla_c(g^{ab}\delta\Gamma^c_{ab}) - \nabla_b(g^{ab}\delta\Gamma^c_{ac})]dv \\
&= \int_D \sqrt{-|g|}\nabla_c(g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab})dv
\end{aligned}$$

This equation is the integral over a volume element of the covariant derivative of a tensor field. By Stokes' Theorem, the above integral vanishes, and we see that δI_1 contributes nothing to the total variation δI.

Next, consider the variation of the metric g_{ab}. Let A^{ab} be the associated co-factor of the metric. Let us fix a, and expand the determinant $|g|$ by the ath row. Then,

$$|g| = |g|_{ab}A^{ab} \tag{51}$$

Taking the partial derivative of $|g|$ with respect to g_{ab}, we have,

$$\frac{\partial |g|}{\partial g_{ab}} = A^{ab} \tag{52}$$

Variation of the determinant $|g|$ is then given by,

$$\begin{aligned} \delta|g| &= \frac{\partial |g|}{\partial g_{ab}} \delta g_{ab} \\ &= A^{ab} \delta g_{ab} \\ &= g g^{ab} \delta g_{ab} \end{aligned}$$

Using the relation above, we have,

$$\begin{aligned} \delta\sqrt{-|g|} &= -\frac{1}{2\sqrt{-|g|}} \delta g \\ &= \frac{1}{2} \frac{g}{\sqrt{-|g|}} g^{ab} \delta g_{ab} \end{aligned}$$

Next, we convert δg_{ab} to δg^{ab} by considering,

$$\begin{aligned} \delta\delta^a_d &= \delta(g_{ac} g^{cd}) = 0 \\ (\delta g_{ac}) g^{cd} + g_{ac}(\delta g^{cd}) &= 0 \\ (\delta g_{ac}) g^{cd} &= -g_{ac}(\delta g^{cd}) \end{aligned}$$

Multiplying both sides by g_{bd},

$$\begin{aligned} g_{bd} g^{cd}(\delta g_{ac}) &= -g_{bd} g_{ac}(\delta g^{cd}) \\ \delta^b_c(\delta g_{ac}) &= -g_{bd} g_{ac}(\delta g^{cd}) \\ \delta g_{ab} &= -g_{bd} g_{ac}(\delta g^{dc}) \end{aligned}$$

Now, we substitute this into the equation for the variation, of $\sqrt{-|g|}$,

$$\begin{aligned} \delta\sqrt{-|g|} &= -\frac{1}{2}\sqrt{-|g|} g^{ab} g_{bd} g_{ac}(\delta g^{dc}) \\ &= -\frac{1}{2}\sqrt{-|g|}\delta^a_d g_{ac}(\delta g^{dc}) \\ &= -\frac{1}{2}\sqrt{-|g|} g_{cd}(\delta g^{dc}) \end{aligned}$$

Renaming letters c to a and d to b, we have,

$$\delta\sqrt{-|g|} = -\frac{1}{2}\sqrt{-g} g_{ab}(\delta g^{ab})$$

Thus, the total variation of the Einstein-Hilbert action is,

$$
\begin{aligned}
\delta I &= \delta(\int_D \sqrt{-|g|}(R-2\Lambda)dv) \\
&= \int_D \sqrt{-|g|}g^{ab}(\delta R_{ab})dv + \int_D \sqrt{-|g|}R_{ab}(\delta g^{ab})dv + \int_D (R-2\Lambda)(\delta\sqrt{-|g|})dv \\
&= \int_D \sqrt{-|g|}R_{ab}(\delta g^{ab})dv - \frac{1}{2}\int_D (R-2\Lambda)\sqrt{-|g|}g_{ab}(\delta g^{ab})dv \\
&= \int_D \sqrt{-|g|}(\delta g^{ab})[R_{ab} - \frac{1}{2}(R-2\Lambda)g_{ab}]dv
\end{aligned}
$$

Stationary points are exactly when $R_{ab} - \frac{1}{2}(R-2\Lambda)g_{ab} = 0$, or when,

$$\frac{1}{\sqrt{-g}}\frac{\delta I}{g_{ab}} = R_{ab} - \frac{1}{2}(R-2\Lambda)g_{ab} = 0$$

Thus, the above calculations derive Einstein's equations in a vacuum, when the energy momentum tensor $T_{ab} \equiv 0$. The full field equations with non-trivial stress-energy tensor or with matter field present can be derived by assuming there is an extra term in the action i.e. the full action S would be given by,

$$S = KI + I_{matter}$$

where K is some suitable constant. If we again vary the metric g_{ab} and divide by the metric determinant, we arrive at a familiar expression,

$$\frac{1}{\sqrt{-|g|}}\frac{\delta S}{\delta g_{ab}} = \frac{1}{\sqrt{-|g|}}(K\frac{\delta I}{\delta g_{ab}} + \frac{\delta I_{matter}}{\delta g_{ab}}) = 0$$

We define the energy momentum tensor to be,

$$T_{ab} \triangleq -2\frac{1}{\sqrt{-|g|}}\frac{I_{matter}}{\delta g^{ab}}$$

From above, we have,

$$
\begin{aligned}
K\frac{1}{\sqrt{-g}}\frac{\delta I}{\delta g_{ab}} &= -\frac{1}{\sqrt{-g}}\frac{\delta I_{matter}}{\delta g_{ab}} \\
K(R_{ab} - \frac{1}{2}(R-2\Lambda)g_{ab}) &= \frac{1}{2}T_{ab}
\end{aligned}
$$

From Newtonian theory, the constant K should be $\frac{1}{16\pi}$ (assuming the gravitational constant G and speed of light c are 1). Thus, we have the full Einstein field equations with non-trivial energy-momentum tensor,

$$R_{ab} - \frac{1}{2}(R-2\Lambda)g_{ab} = 8\pi T_{ab}$$

4 Exact Solutions

An exact solution is a spacetime (M, g) for which the Einstein field equations are satisfied under certain assumptions. As the Einstein equations are effectively a system of six non-linear partial differential equations, the presented complexity makes exact solutions difficult to find unless the space-time is assumed beforehand to have certain symmetries. However, exact solutions have been obtained for various fields including for a vacuum ($T_{ab} \equiv 0$), an electromagnetic field, perfect fluids of pressure p and density μ, and others. In particular, the theory of black holes is rooted deeply in spacetimes resulting from a massive body. Once the spacetimes are found, in order to analyze their underlying geometry, a nice choice of coordinates for the metric must be made, since poor coordinates give the appearance that the metric has singularities when no singularities actually exist. However, under ideal coordinate transforms, it becomes much easier to study and visualize the spacetime geometry, such as the behavior of geodesics. It suffices to study the geometry of a metric $\bar{g}$ conformal to the original spacetime metric g of interest, since the null cone structure is preserved i.e.

$$g(X, X) > 0, = 0, < 0 \Rightarrow \bar{g}(X, X) > 0, = 0, < 0$$

Null geodesics in a spacetime have the same image in a conformal spacetime. Thus, we shall try to make the conformal metric $\bar{g}$ as simple as possible. In each of the ensuing spacetimes described, we make the necessary coordinate transformations that simplify our spacetime in order to analyze their geometry. One of the diagrams we use in our descriptions will be Penrose diagrams, which provide a way to visualize spherically symmetric spacetimes.

4.1 Minkowski spacetime

The first spacetime we shall study is Minkowski space-time (M, g), since it is the simplest space-time resulting from the field equations. It is the space-time of interest in special relativity, which replaces Newtonian absolute space $\mathbb{R}^3$ and absolute time $\mathbb{R}$ by a 3+1-dimensional manifold $\mathbb{R}^3 \times \mathbb{R}$. Due to the simplicity of the metric, it can be easily verified that the metric indeed satisfies the field equations; the Riemann tensor will vanish, and so do the Ricci tensor and scalar curvature. Therefore, Minkowski space-time is flat with constant curvature of zero. In the natural coordinates (x^1, x^2, x^3, x^4), the metric g can be expressed in the form

$$ds^2 = -(dx^4)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{53}$$

We take time t to be the coordinate variable x^4, and space to be the other three. Transforming to spherical coordinates (t, r, θ, ϕ) which are defined in the ranges $0 < r < \infty$, $0 < \theta < \phi$, and $0 < \phi < 2\pi$, let $x^4 = t$, $x^3 = r\cos\theta$,

$x^2 = r\sin\theta\cos\phi$, and $x^1 = r\sin\theta\sin\phi$. Then,

$$\begin{aligned}
dx^1 &= \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi \\
dx^2 &= \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi \\
dx^3 &= \cos\theta dr - r\sin\theta d\theta \\
dx^4 &= dt
\end{aligned}$$

Plugging these transformations into (53), we arrive at,

$$ds^2 = -(dt)^2 + (dr)^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \tag{54}$$

In this form, the metric appears to be singular at $r = 0$ and $\sin\theta = 0$. However, it is in fact not singular since spherical coordinates are defined in domains of r, θ to disallow this possibility.

We can also choose advanced and retarded coordinates v, w defined by $v = t + r$ and $w = t - r$. Under this coordinate system,

$$\begin{aligned}
dv &= dt + dr \\
dw &= dt - dr
\end{aligned}$$

Since $-dvdw = -dt^2 + dr^2$, the metric becomes,

$$ds^2 = -dvdw + \frac{1}{4}(v-w)^2((d\theta)^2 + sin^2\theta(d\phi)^2) \tag{55}$$

with $-\infty < v, w < \infty$.

As the metric is flat, the connection coefficients $\Gamma^\mu_{\lambda\nu}$ vanish. The geodesic equation in this setting will be,

$$\begin{aligned}
\frac{\partial^2 x^\mu}{ds^2} + \Gamma^\mu_{\lambda\nu}\frac{\partial x^\lambda}{ds}\frac{\partial x^\nu}{ds} &= 0 \\
\Rightarrow \frac{d^2 x^\mu}{ds^2} &= 0
\end{aligned}$$

where x^μ are the components of the four vector, and s is the associated affine parameter of the curve. Integrating twice, we see that geodesics in Minkowski space will simply be straight lines of the form $x^\mu(s) = a^\mu s + b^\mu$. The exponential map $exp_p : T_p \to M$ is thus defined as $x^\mu(exp_p X) = X^\mu + a^\mu s + b^\mu$, where $X \in T_p$ is expressed in the coordinate basis $\{\frac{\partial}{\partial x^\mu}\}$. This linear map is one to one and onto, and hence it is a diffeomorphism between T_p and M, as any two points in Minkowski space can be joined by a unique geodesic. This implies that the exponential map is defined everywhere, so Minkowski space is geodesically complete.

The surfaces $\{x^4 = constant\}$ are Cauchy surfaces which cover all of M. This is easy to see geometrically, since Minkowski space can be visualized as $\mathbb{R}^3 \times \mathbb{R}$. Every inextendible, causal curve intersects the surfaces $\{x^4 = constant\}$, which implies that Minkowski space is globally hyperbolic, since there exists at least one Cauchy surface. However, it is easy to also find non-Cauchy surfaces. Take the surface $S_\sigma = \{-(x^4)^2+(x^3)^2+(x^2)^2+(x^1)^2 = \sigma = constant\}$ with $\sigma, x^4 < 0$. Since $\sigma < 0$, this surface will be completely contained in the past null cone of the origin O. As such, if one takes a inextendible timelike curve that does not intersect the past null cone of the origin, then this curve will not intersect S_σ.

To study the structure of infinity in Minkowski spacetime, we compactify our space such that singularities of the metric are reached at finite values. This will allow us to study the spacetime within some finite region. From the advanced and retarded coordinates (55), we can further define p, q such that $\tan p = v$, $\tan q = w$ where $-\frac{\pi}{2} < p < \frac{\pi}{2}$ and $-\frac{\pi}{2} < q < \frac{\pi}{2}$. The metric then transforms to,

$$ds^2 = \sec^2 p \sec^2 q(-dpdq + \frac{1}{4}\sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2)) \tag{56}$$

which is seen to be conformal to the metric,

$$d\bar{s}^2 = -4dpdq + \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2) \tag{57}$$

since,

$$ds^2 = \frac{1}{4}\sec^2 p \sec^2 q d\bar{s}^2 \tag{58}$$

From this, if we let $t' = p + q$ and $r' = p - q$, this transforms the metric $d\bar{s}^2$ to,

$$d\bar{s}^2 = -(dt')^2 + (dr')^2 + \sin^2(r')(d\theta^2 + \sin^2\theta d\phi^2) \tag{59}$$

Thus, for,

$$-\pi < t' + r' < \pi, \qquad -\pi < t' - r' < \pi, \qquad r' \geq 0$$

the metric (56) becomes,

$$ds^2 = \frac{1}{4}\sec^2(\frac{1}{2}(t'+r'))\sec^2(\frac{1}{2}(t'-r'))d\bar{s}^2 \tag{60}$$

As shown by the metric (60), the whole of Minkowski space is in fact conformal to a region of the Einstein static universe, which is a cosmological model of the universe under assumptions that the universe is spatially homogeneous. It is given by the four dimensional manifold $\mathbb{R} \times \mathbb{S}^3$ embedded in five dimensional Minkowski space. If we suppress two dimensions, it can be represented by a cylinder $x^2 + y^2 = 1$ embedded in three dimensional Minkowski space.

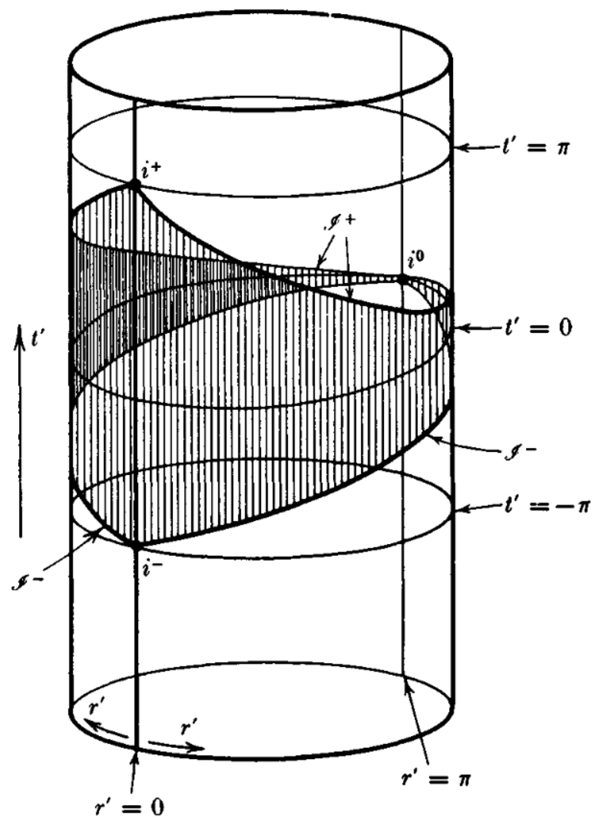

Figure 1: Minkowski space embedded in the Einstein static universe with two dimensions suppressed. [3], pg.122

The shaded region is conformally equivalent to Minkowski space. The boundary of this region consists of surfaces S^+ $(p = \frac{\pi}{2})$ and S^- $(q = -\frac{\pi}{2})$, and points i^+ $(p = \frac{\pi}{2}, q = \frac{\pi}{2})$, i^0 $(p = \frac{\pi}{2}, q = -\frac{\pi}{2})$, and i^- $(p = -\frac{\pi}{2}, q = -\frac{\pi}{2})$ with respect to the metric (60). The Penrose diagram of Minkowski space is given below.

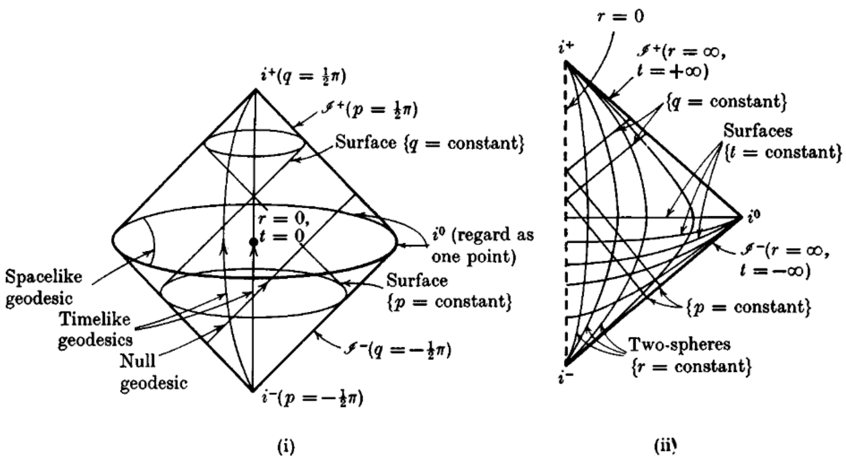

Figure 2: (i) Region representing Minkowski space-time and its conformal infinity. (ii) Penrose diagram of Minkowski space. [3], pg.123

We can now examine the behavior of geodesics in Minkowski space. i^+ (i^-) is a future (past) timelike infinity, since all timelike geodesics will eventually reach one of these two points depending on whether they are future- or past-directed. Similarly, the surfaces S^+ and S^- are future and past null infinities respectively, since one can regard null geodesics as originating from S^- and going to S^+. The point i^0 is itself a future and past spacelike infinity, since spacelike geodesics will originate and end at i^0. Note that this behavior only applies to geodesics as a non-geodesic timelike curve may originate and end in a different manner.

4.2 De Sitter spacetime

Like Minkowski space-time, de Sitter space-time is another solution of the vacuum field equations, but instead has constant curvature $R > 0$. It is the solution when $\Lambda = \frac{R}{4}$, where the Einstein equations take the form,

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\frac{1}{4}Rg_{ab} \tag{61}$$

The positive cosmological constant represents an expanding universe. It has topology $\mathbb{R}^1 \times S^3$ and can be visualized as the hyperboloid,

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \tag{62}$$

embedded in $\mathbb{R}^5$ with the familiar Lorentzian metric of signature three,

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2 \tag{63}$$

We make the first coordinate change by defining variables (t, χ, θ, ϕ),

$$\begin{aligned}
\alpha \sinh(\alpha^{-1} t) &= v \\
\alpha \cosh(\alpha^{-1} t) \cos(\chi) &= w \\
\alpha \cosh(\alpha^{-1} t) \sin(\chi) \cos(\theta) &= x \\
\alpha \cosh(\alpha^{-1} t) \sin(\chi) \sin(\theta) \cos(\phi) &= y \\
\alpha \cosh(\alpha^{-1} t) \sin(\chi) \sin(\theta) \sin(\phi) &= z
\end{aligned}$$

Plugging in the following expressions into the metric (63),

$$\begin{aligned}
dv &= \cosh(\alpha^{-1} t) dt \\
dw &= \sinh(\alpha^{-1} t) \cos(\chi) dt - \alpha \cosh(\alpha^{-1} t) \sin(\chi) d\chi \\
dx &= f + e \\
dy &= c + d \\
dz &= a + b
\end{aligned}$$

where,

$$\begin{aligned}
a &= \sinh(\alpha^{-1} t) \sin(\chi) \sin(\theta) dt + \alpha \cosh(\alpha^{-1} t) \cos(\chi) \sin(\theta) \sin(\phi) d\chi \\
b &= \alpha \cosh(\alpha^{-1} t) \sin(\chi) \cos(\theta) \sin(\phi) d\theta + \alpha \cosh(\alpha^{-1} t) \sin(\chi) \sin(\theta) \cos(\phi) d\phi \\
c &= \sinh(\alpha^{-1} t) \sin(\chi) \sin(\theta) \cos(\phi) dt + \alpha \cosh(\alpha^{-1} t) \cos(\chi) \sin(\theta) \cos(\phi) d\chi \\
d &= \alpha \cosh(\alpha^{-1} t) \sin(\chi) \cos(\theta) \cos(\phi) d\theta - \alpha \cosh(\alpha^{-1} t) \sin(\chi) \sin(\theta) \sin(\phi) d\phi \\
e &= \alpha \cosh(\alpha^{-1} t) \cos(\chi) \cos(\theta) d\chi - \alpha \cosh(\alpha^{-1} t) \sin(\chi) \sin(\theta) d\theta \\
f &= \sinh(\alpha^{-1} t) \sin(\chi) \cos(\theta) dt
\end{aligned}$$

the metric transforms to,

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(\alpha^{-1} t)[d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{64}$$

The apparent singularities of this metric $\theta = 0, \pi$ and $\chi = 0, \pi$ are in fact trivial due to the choice of polar coordinates. These coordinates cover the whole hyperboloid for $-\infty < t < \infty, 0 \le \chi \le \pi, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$. If one also introduces the coordinates,

$$\begin{aligned}
\bar{t} &= \alpha \log(\frac{w + v}{\alpha}) \\
\bar{x} &= \frac{\alpha x}{w + v} \\
\bar{y} &= \frac{\alpha y}{w + v} \\
\bar{z} &= \frac{\alpha z}{w + v}
\end{aligned}$$

then the metric takes the form,

$$ds^2 = -d\bar{t}^2 + exp(2\alpha^{-1}\bar{t})(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2) \tag{65}$$

These coordinates only cover half of the embedded hyperboloid, since the logarithm and thus $\bar{t}$ is not defined for $w + v \le 0$.

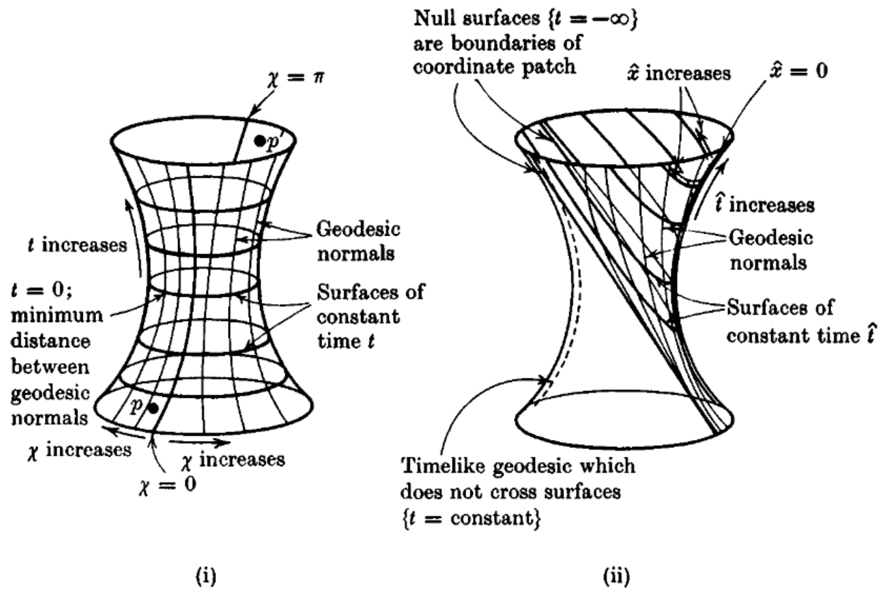

Figure 3: De Sitter space represented by a hyperboloid embedded in five dimensional Minkowski space. (i) The coordinates (t, χ, θ, ϕ) cover the whole hyperboloid. (ii) The coordinates $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ cover half of the hyperboloid. [3], pg.125

To study the structure of conformal infinity of de-Sitter space, we once again transform the metric into one that is conformal to the metric of the Einstein static universe (59). Just as with Minkowski space, this will allow us to more easily visualize the geometry when the space is embedded in the static cylinder. We make another coordinate change, defining a new time variable t',

$$t' = 2\arctan(\exp(\alpha^{-1}t)) - \frac{\pi}{2} \tag{66}$$

where $\frac{-\pi}{2} < t' < \frac{\pi}{2}$. This transforms the metric to,

$$ds^2 = \alpha^2 \cosh^2(\alpha^{-1}t')d\bar{s}^2 \tag{67}$$

where $d\bar{s}$ is the metric (59).

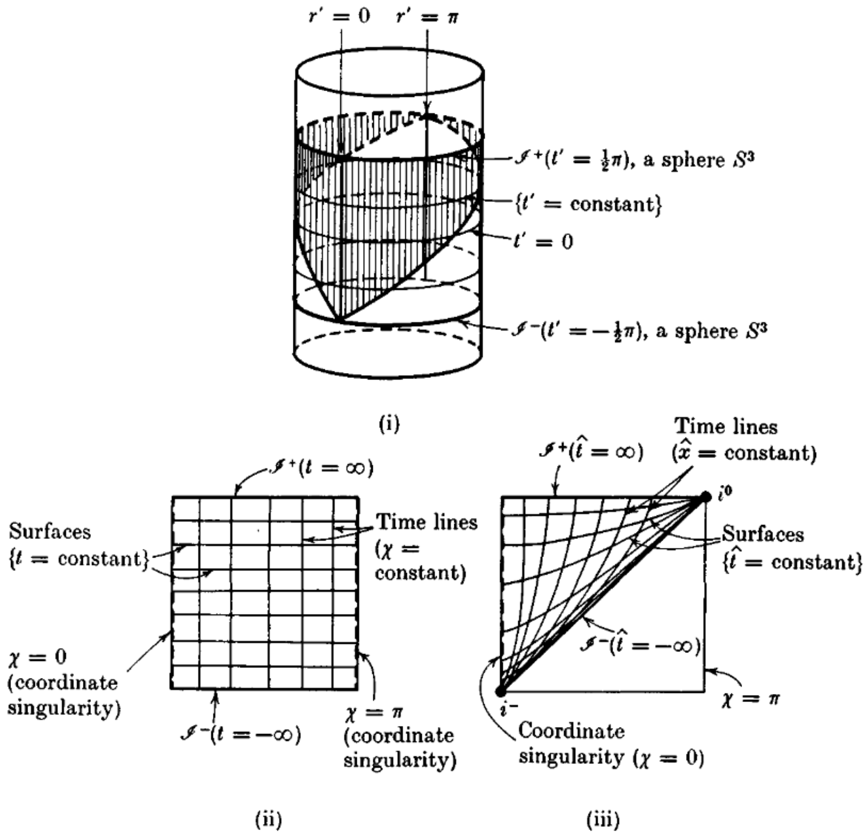

Figure 4: (i) de Sitter space embedded in the Einstein static universe. (ii) Penrose diagram for the full de Sitter space under coordinates (65). (iii) Penrose diagram for half of de Sitter space under coordinates (67). [3], pg.127

The following figure shows the region of the Einstein static universe in which de Sitter space is embedded as given by the metric (67), as well as the Penrose diagram of de Sitter space. From the Penrose diagram, we see that unlike Minkowski space, de Sitter space has both past and future spacelike infinities for timelike and null curves. Because of this, de Sitter space has a much different causal structure than Minkowski space. Recall that one of the central postulates of relativity is that information cannot travel faster than light, which travels on null geodesics. Given an observer O and any point p on its world-line, we can visualize the limit of the set of events that can influence O at p, and the set of events O can influence from p by taking forwards and backwards light cones from p. To determine the boundary of which events can be observed by O and which events can never be observed by O, we simply draw a backwards light cone from future infinity on the world-line of O. Such a boundary is called the future event horizon. It is easy to see from the Penrose diagrams that if one draws the future event horizon (straight null lines at 45 degrees) in de-Sitter space, then the region representing the set of events that can be observed by O will not be the whole space. However, in Minkowski space, since the infinities themselves are null, the future event horizon will indeed cover the whole space. Therefore, this means that for every event in Minkowski space there will always exist a point p along O's worldline in which the backwards light cone includes that event.

Expanding upon these considerations, because there can exist future event horizons in de Sitter space, given two observers O_1 and O_2 and their worldlines, there will be a limit to the events along O_2's worldline that can be observed by O_1. This means though an infinite amount of time passes for O_1 to reach spacelike infinity, O_1 will only ever observe a finite amount of O_2's history.

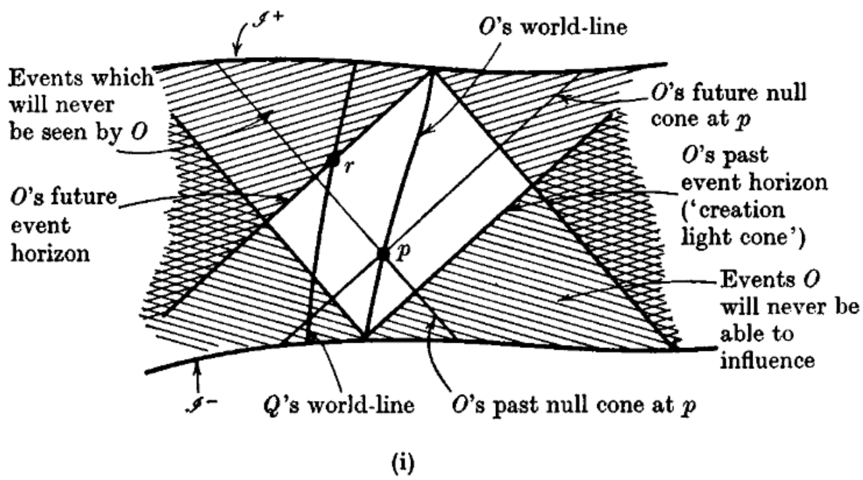

Figure 5: The future event horizon has a non-trivial existence in de-Sitter space that gives arise to a different causal structure than that of Minkowski space. [3], pg.130

4.3 Anti-de Sitter spacetime

The third vacuum solution to the Einstein equations we shall discuss is anti-de Sitter spacetime. As the name suggests, this spacetime has constant curvature $R < 0$ and a topology of $\mathbb{R}^1 \times S^3$. One takes the cosmological constant to be negative, which represents a contracting universe. Like de-Sitter space, it can be represented by the hyperboloid ,

$$-v^2 - u^2 + x^2 + y^2 + z^2 = \alpha^2 \tag{68}$$

embdeed in $\mathbb{R}^5$ with the metric,

$$ds^2 = -(dv)^2 - (du)^2 + (dx)^2 + (dy)^2 + (dz)^2 \tag{69}$$

One of the key things to note is that there exist closed timelike curves in this space under the metric (69) e.g. the curve parametrized by $v = \alpha\cos(t)$ and $u = \alpha\sin(t)$ with all other coordinates zero is a closed timelike curve. To avoid this, we must instead study the covering space of anti-de Sitter space, which has the topology of $\mathbb{R}^4$ (since the covering space of $\mathbb{S}^1$ is $\mathbb{R}$). This space has no closed timelike curves. The metric of the universal covering space is given by,

$$ds^2 = -dt^2 + \cos^2 t d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \tag{70}$$

Despite the fact that this metric only covers half of anti de-Sitter space, and has apparent singularities at $t = \pm\frac{\pi}{2}$, we can avoid this by defining another set of coordinates (t', r, θ, ϕ) to get,

$$ds^2 = -\cosh^2 r(dt')^2 + dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta d\phi^2) \tag{71}$$

To study the structure of infinity in anti de-Sitter space, we make the change,

$$r' = 2\arctan(\exp r) - \frac{\pi}{2} \tag{72}$$

for $0 \leq r' \leq \frac{\pi}{2}$. The metric will then be conformal to that of the Einstein static universe,

$$ds^2 = \cosh^2 r d\bar{s}^2 \tag{73}$$

where $d\bar{s}^2$ is again given by (59), and covers half of the static universe.

Anti de-Sitter space has an interesting geometry of geodesics. Conformal timelike infinity consists of a timelike surface S and two disjoint points i^+ and i^-. The lines where $\{\chi, \theta, \phi = constant\}$ are timelike geodesics orthogonal to the surfaces $\{t = constant\}$. These geodesics converge at points p and q, and then diverge again into similar diamond shaped regions. They progressively expand out from p, but never reach the timelike surface S.

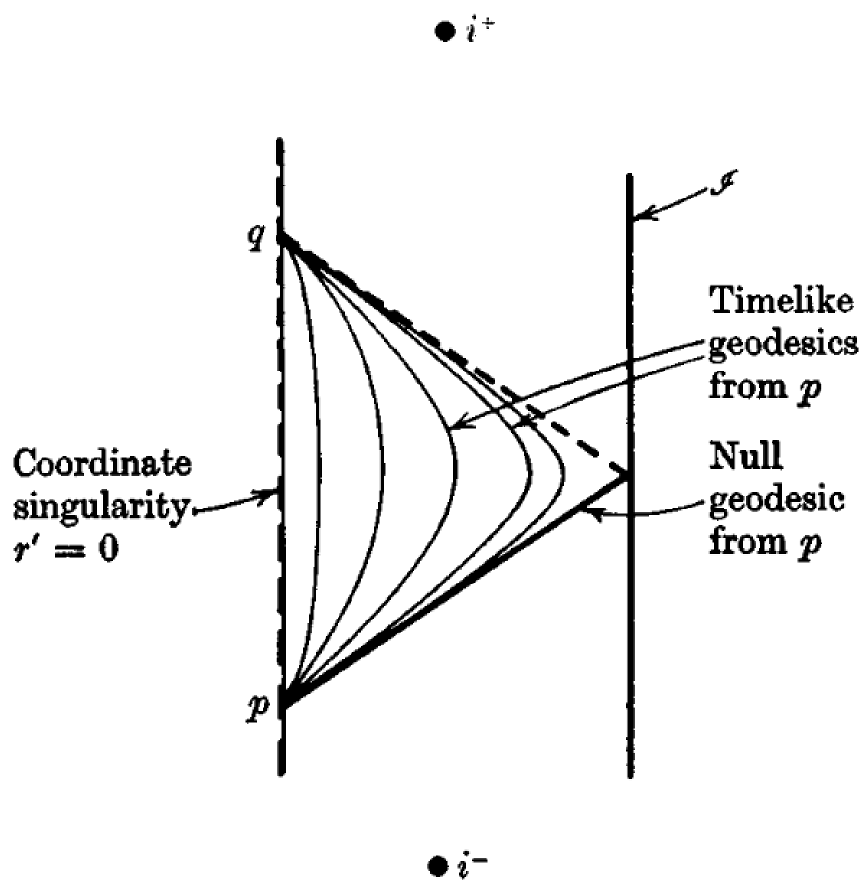

Figure 6: Projected timelike and null geodesics in the region covered by the coordinates (t, χ, θ, ϕ) (73). [3], pg.132

The set of points that can be reached from p by future directed timelike geodesics is the interior of an infinite chain of diamond-shaped regions similar to that covered by coordinates (t, χ, θ, ϕ). However, there exist regions in the future of p that cannot be reached from by any geodesic, but simply by future directed timelike curves.

Furthermore, there exists no Cauchy surface in anti de-Sitter space. For example, if one takes the set of surfaces $\{t' = constant\}$ which cover the whole space, there will be null geodesics that never intersect a surface in the family, as shown in Figure 7. Thus, given initial data on any surface in anti de-Sitter space, one can only predict within the Cauchy development of a given region.

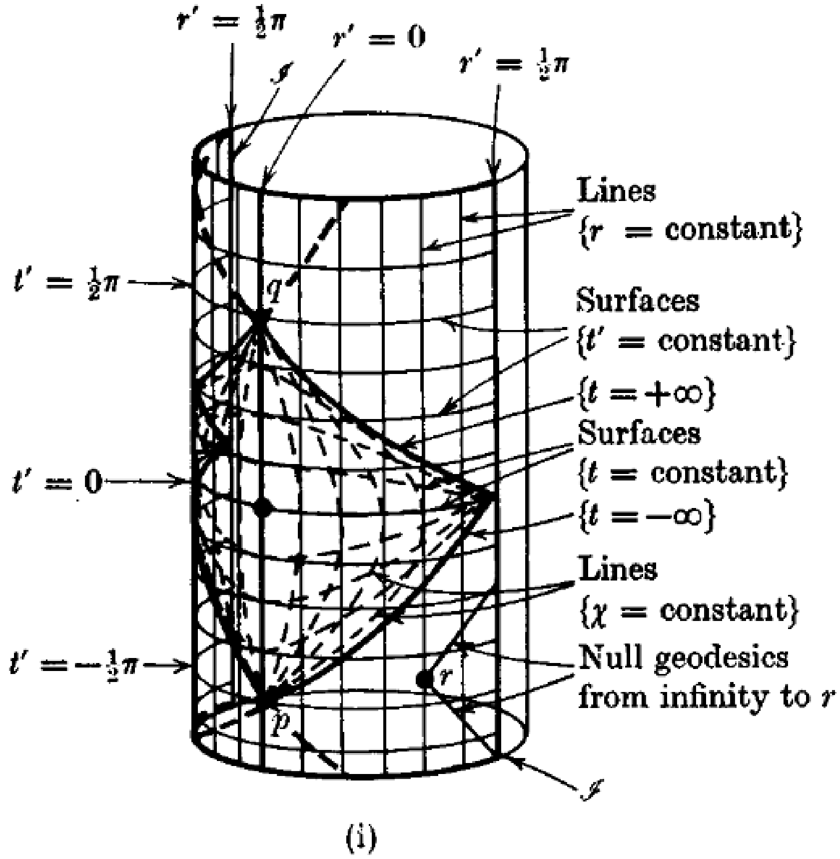

Figure 7: Embedding of anti de-Sitter space in the Einstein static universe. [3], pg.132

4.4 Schwarzchild spacetime

The Schwarzchild space-time was a solution discovered shortly after Einstein put forth his field equations, and describes spacetime near a spherically symmetric massive body of radius r and mass m. To derive the metric, we make a couple of assumptions, namely that the metric is static i.e. the components of g_{ab} do not depend on t, and spherically symmetric i.e. they do not depend on angles ϕ and θ. Furthermore, we want our metric to be Minkowskian in the limit as the mass $m \to 0$, since the spacetime would just be flat given by the vacuum field equations. We label our indices from 0 to 3, letting $(0 = t, 1 = r, 2 = \theta, 3 = \phi)$. Therefore under these assumptions, we want our metric to take the general form in spherical coordinates,

$$ds^2 = \sum_{\mu\eta} g_{\mu\nu} dx^\mu dx^\nu = -U(r)dt^2 + V(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{74}$$

or correspondingly the components of the metric are,

$$\begin{aligned} g_{00} &= -U(r) \\ g_{11} &= V(r) \\ g_{22} &= r^2 \\ g_{33} &= r^2 \sin^2\theta \end{aligned}$$

Notice that U and V only depend on r, and hence $\partial_i U$ and $\partial_i V$ will be zero for $i \neq 1$. As the metric is diagonal, the components of the inverse metric g^{ab} are easily seen to be,

$$\begin{aligned} g^{00} &= \frac{-1}{U(r)} \\ g^{11} &= \frac{1}{V(r)} \\ g^{22} &= \frac{1}{r^2} \\ g^{33} &= \frac{1}{r^2 \sin^2\theta} \end{aligned}$$

We substitute this general form into the field equations to find the functions $U(r)$ and $V(r)$. To do so, we first calculate the Christoffel symbols,

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda}(g_{\lambda\nu,\sigma} + g_{\lambda\sigma,\nu} - g_{\nu\sigma,\lambda}) \tag{75}$$

The non-vanishing Christoffel symbols will be,

$$\begin{aligned} \Gamma^0_{01} &= \frac{1}{2} g^{00}[g_{00,1} + g_{01,0} - g_{01,0}] \\ &= \frac{1}{2U}\partial_r U \\ \Gamma^1_{00} &= \frac{1}{2} g^{11}[g_{10,0} + g_{10,0} - g_{00,1}] \\ &= \frac{1}{2} g^{11} g_{00,1} \\ &= \frac{1}{2V}\partial_r U \\ \Gamma^1_{11} &= \frac{1}{2} g^{11}[g_{11,1} + g_{11,1} - g_{11,1}] \\ &= \frac{1}{2} g^{11} g_{11,1} \\ &= \frac{1}{2V}\partial_r V \end{aligned}$$

$$
\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}[g_{12,2} + g_{12,2} - g_{22,1}] \\
&= \frac{-1}{2}g^{11}g_{22,1} \\
&= \frac{-1}{V}r \\
\Gamma^1_{33} &= \frac{1}{2}g^{11}[g_{13,3} + g_{13,3} - g_{33,1}] \\
&= \frac{-1}{2}g^{11}g_{33,1} \\
&= \frac{-r}{V}\sin^2\theta \\
\Gamma^2_{33} &= \frac{1}{2}g^{22}[g_{23,3} + g_{23,3} - g_{33,2}] \\
&= \frac{-1}{2}g^{22}g_{33,2} \\
&= \frac{-1}{2r^2}r^2 2\sin\theta\cos\theta \\
&= -\sin\theta\cos\theta \\
\Gamma^2_{12} &= \frac{1}{2}g^{22}[g_{21,2} + g_{22,1} - g_{12,2}] \\
&= \frac{1}{2}g^{22}g_{22,1} \\
&= \frac{1}{r} \\
\Gamma^3_{13} &= \frac{1}{2}g^{33}[g_{31,3} + g_{33,1}] \\
&= \frac{1}{2}g^{33}g_{33,1} \\
&= \frac{1}{2r^2\sin^2\theta}2r\sin^2\theta \\
&= \frac{1}{r} \\
\Gamma^3_{23} &= \frac{1}{2}g^{33}[g_{32,3} + g_{33,2}] \\
&= \frac{1}{2}g^{33}g_{33,2} \\
&= \frac{1}{2r^2\sin^2\theta}r^2 2\sin\theta\cos\theta \\
&= \cot\theta
\end{aligned}
$$

Since the Christoffel symbols are symmetric in the lower two indices, we actually have twice the number of quantities. We can now calculate the components of the Ricci tensor, which is the contraction of the Riemann curvature tensor,

$$R_{\mu\nu} = R^{\beta}_{\mu\nu\beta} = \Gamma^{\beta}_{\mu\beta,\nu} - \Gamma^{\beta}_{\mu\nu,\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} \tag{76}$$

It can be shown that $R_{\mu\nu} = 0$ for $\mu \neq \nu$. Hence, we calculate $R_{\mu\nu}$ when $\mu = \nu$,

$$\begin{aligned}
R_{00} = {} & \Gamma^{0}_{00,0} - \Gamma^{0}_{00,0} + \Gamma^{\alpha}_{00}\Gamma^{0}_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^{0}_{\alpha 0} \\
& + \Gamma^{1}_{01,0} - \Gamma^{1}_{00,1} + \Gamma^{\alpha}_{01}\Gamma^{1}_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^{1}_{\alpha 1} \\
& + \Gamma^{2}_{02,0} - \Gamma^{2}_{00,2} + \Gamma^{\alpha}_{02}\Gamma^{2}_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^{2}_{\alpha 2} \\
& + \Gamma^{3}_{03,0} - \Gamma^{3}_{00,3} + \Gamma^{\alpha}_{03}\Gamma^{3}_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^{3}_{\alpha 3} \\
& = (cancel pairwise) \\
& + 0 - \Gamma^{1}_{00,1} + \Gamma^{0}_{01}\Gamma^{1}_{00} - \Gamma^{1}_{11}\Gamma^{1}_{00} \\
& + 0 - 0 + 0 - \Gamma^{1}_{00}\Gamma^{2}_{12} \\
& + 0 - 0 + 0 - \Gamma^{1}_{00}\Gamma^{3}_{13}
\end{aligned}$$

This implies that,

$$\begin{aligned}
R_{00} &= -\Gamma^{1}_{00,1} + \Gamma^{0}_{01}\Gamma^{1}_{00} - \Gamma^{1}_{11}\Gamma^{1}_{00} - \Gamma^{1}_{00}\Gamma^{2}_{12} - \Gamma^{1}_{00}\Gamma^{3}_{13} \\
&= -\partial_r\left(\frac{\partial_r U}{2V}\right) + \frac{\partial_r U}{2U}\frac{\partial_r U}{2V} - \frac{\partial_r V}{2V}\frac{\partial_r U}{2V} - \frac{\partial_r U}{2V}\frac{1}{r} - \frac{\partial_r U}{2V}\frac{1}{r} \\
&= \frac{-\partial_r^2 U}{2V} + \frac{\partial_r U \partial_r V}{2V^2} + \frac{(\partial_r U)^2}{4UV} - \frac{\partial_r U \partial_r V}{4V^2} - \frac{\partial_r U}{Vr} \\
&= \frac{-\partial_r^2 U}{2V} + \frac{\partial_r U \partial_r V}{4V^2} + \frac{(\partial_r U)^2}{4UV} - \frac{\partial_r U}{Vr}
\end{aligned}$$

Similarly, for R_{11},

$$\begin{aligned}
R_{11} = {} & \Gamma^{0}_{10,1} - \Gamma^{0}_{11,0} + \Gamma^{\alpha}_{10}\Gamma^{0}_{\alpha 1} - \Gamma^{\alpha}_{11}\Gamma^{0}_{\alpha 0} \\
& + \Gamma^{1}_{11,1} - \Gamma^{1}_{11,1} + \Gamma^{\alpha}_{11}\Gamma^{1}_{\alpha 1} - \Gamma^{\alpha}_{11}\Gamma^{1}_{\alpha 1} \\
& + \Gamma^{2}_{12,1} - \Gamma^{2}_{11,2} + \Gamma^{\alpha}_{12}\Gamma^{2}_{\alpha 1} - \Gamma^{\alpha}_{11}\Gamma^{2}_{\alpha 2} \\
& + \Gamma^{3}_{13,1} - \Gamma^{3}_{11,3} + \Gamma^{\alpha}_{13}\Gamma^{3}_{\alpha 1} - \Gamma^{\alpha}_{11}\Gamma^{3}_{\alpha 3} \\
& = \Gamma^{0}_{10,1} - 0 + \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{1}_{11}\Gamma^{0}_{10} \\
& + (cancel pairwise) \\
& + \Gamma^{2}_{12,1} - 0 + \Gamma^{2}_{12}\Gamma^{2}_{21} - \Gamma^{1}_{11}\Gamma^{2}_{12} \\
& + \Gamma^{3}_{13,1} - 0 + \Gamma^{3}_{13}\Gamma^{3}_{31} - \Gamma^{1}_{11}\Gamma^{3}_{13}
\end{aligned}$$

Hence,

$$\begin{aligned}
R_{11} &= \Gamma^0_{10,1} + \Gamma^0_{10}\Gamma^0_{01} - \Gamma^1_{11}\Gamma^0_{10} + \Gamma^2_{12,1} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^1_{11}\Gamma^2_{12} \\
&+ \Gamma^3_{13,1} + \Gamma^3_{13}\Gamma^3_{31} - \Gamma^1_{11}\Gamma^3_{13} \\
&= \partial_r(\frac{\partial_r U}{2U}) + (\frac{\partial_r U}{2U})^2 - \frac{\partial_r V \partial_r U}{4UV} - \frac{1}{r^2} + \frac{1}{r^2} - \frac{\partial_r V}{2Vr} - \frac{1}{r^2} + \frac{1}{r^2} - \frac{\partial_r V}{2Vr} \\
&= \frac{\partial_r^2 U}{2U} - \frac{(\partial_r U)^2}{2U^2} + \frac{(\partial_r U)^2}{4U^2} - \frac{\partial_r U \partial_r V}{4UV} - \frac{\partial_r V}{Vr} \\
&= \frac{\partial_r^2 U}{2U} - \frac{(\partial_r U)^2}{4U^2} - \frac{\partial_r U \partial_r V}{4UV} - \frac{\partial_r V}{Vr}
\end{aligned}$$

Now, for R_{22},

$$\begin{aligned}
R_{22} &= \Gamma^0_{20,2} - \Gamma^0_{22,0} + \Gamma^\alpha_{20}\Gamma^0_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^0_{\alpha 0} \\
&+ \Gamma^1_{21,2} - \Gamma^1_{22,1} + \Gamma^\alpha_{21}\Gamma^1_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^1_{\alpha 1} \\
&+ \Gamma^2_{22,2} - \Gamma^2_{22,2} + \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^2_{\alpha 2} \\
&+ \Gamma^3_{23,2} - \Gamma^3_{22,3} + \Gamma^\alpha_{23}\Gamma^3_{\alpha 2} - \Gamma^\alpha_{22}\Gamma^3_{\alpha 3} \\
&= 0 - 0 + 0 - \Gamma^1_{22}\Gamma^0_{10} \\
&+ 0 - \Gamma^1_{22,1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} \\
&+ (cancelpairwise) \\
&+ \Gamma^3_{23,2} - 0 + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13}
\end{aligned}$$

and thus,

$$\begin{aligned}
R_{22} &= -\Gamma^1_{22}\Gamma^0_{10} - \Gamma^1_{22,1} + \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^1_{11} + \Gamma^3_{23,2} + \Gamma^3_{23}\Gamma^3_{32} - \Gamma^1_{22}\Gamma^3_{13} \\
&= \frac{r\partial_r U}{2UV} + \partial_r(\frac{r}{V}) - \frac{1}{V} + \frac{r\partial_r V}{2V^2} + \partial_\theta(\cot\theta) + (\cot\theta)^2 + \frac{1}{V} \\
&= \frac{r\partial_r U}{2UV} + (\frac{1}{V} - \frac{r\partial_r V}{V^2}) - \frac{1}{V} + \frac{r\partial_r V}{2V^2} + (-1 - \cot^2\theta) + (\cot\theta)^2 + \frac{1}{V} \\
&= \frac{r\partial_r U}{2UV} - \frac{r\partial_r V}{V^2} + \frac{r\partial_r V}{2V^2} - 1 + \frac{1}{V} \\
&= \frac{r\partial_r U}{2UV} - \frac{r\partial_r V}{2V^2} - 1 + \frac{1}{V}
\end{aligned}$$

Finally, for R_{33},

$$
\begin{aligned}
R_{33} &= \Gamma^0_{30,3} - \Gamma^0_{33,0} + \Gamma^\alpha_{30}\Gamma^0_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^0_{\alpha 0} \\
&+ \Gamma^1_{31,3} - \Gamma^1_{33,1} + \Gamma^\alpha_{31}\Gamma^1_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^1_{\alpha 1} \\
&+ \Gamma^2_{32,3} - \Gamma^2_{33,2} + \Gamma^\alpha_{32}\Gamma^2_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^2_{\alpha 2} \\
&+ \Gamma^3_{33,3} - \Gamma^3_{33,3} + \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} - \Gamma^\alpha_{33}\Gamma^3_{\alpha 3} \\
&= 0 - 0 + 0 - \Gamma^1_{33}\Gamma^0_{10} \\
&+ 0 - \Gamma^1_{33,1} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^1_{11} \\
&+ 0 - \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^1_{33}\Gamma^2_{12} \\
&+ (cancelpairwise)
\end{aligned}
$$

and we have,

$$
\begin{aligned}
R_{33} &= -\Gamma^1_{33}\Gamma^0_{10} - \Gamma^1_{33,1} + \Gamma^3_{31}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^1_{11} - \Gamma^2_{33,2} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^1_{33}\Gamma^2_{12} \\
&= \frac{r}{V}\sin^2\theta\frac{\partial_r U}{2U} + \partial_r(\frac{r\sin^2\theta}{V}) - \frac{\sin^2\theta}{V} + (\frac{r\sin^2\theta}{V})\frac{\partial_r V}{2V} \\
&+ \partial_\theta(\sin\theta\cos\theta) + \cot\theta(-\sin\theta\cos\theta) + \frac{\sin^2\theta}{V} \\
&= \frac{r}{V}\sin^2\theta\frac{\partial_r U}{2U} + (\frac{\sin\theta}{V} - \frac{r\partial_r V\sin^2\theta}{V^2}) - \frac{\sin^2\theta}{V} + (\frac{r\sin^2\theta}{V})\frac{\partial_r V}{2V} \\
&+ (\cos^2\theta - \sin^2\theta) - \cos^2\theta + \frac{\sin^2\theta}{V} \\
&= (\frac{r}{V}\frac{\partial_r U}{2U} - \frac{r\partial_r V}{2V^2} - 1 + \frac{1}{V})\sin^2\theta \\
&= \sin^2\theta R_{22}
\end{aligned}
$$

With the components of the Ricci tensor, we can now calculate the scalar curvature, which is the contraction of the Ricci tensor,

$$
R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \tag{77}
$$

We have as the scalar curvature,

$$
\begin{aligned}
R &= \frac{-1}{U}(\frac{-\partial_r^2 U}{2V} + \frac{\partial_r U\partial_r V}{4V^2} + \frac{(\partial_r U)^2}{4UV} - \frac{\partial_r U}{Vr}) + \frac{1}{V}(\frac{\partial_r^2 U}{2U} - \frac{(\partial_r U)^2}{4U^2} \\
&- \frac{\partial_r U\partial_r V}{4UV} - \frac{\partial_r V}{Vr}) + \frac{1}{r^2}(\frac{r\partial_r U}{2UV} - \frac{r\partial_r V}{2V^2} - 1 + \frac{1}{V}) + \frac{1}{r^2\sin^2\theta}(\sin^2\theta R_{22}) \\
&= \frac{\partial_r^2 U}{2UV} - \frac{\partial_r U\partial_r V}{4UV^2} - \frac{(\partial_r U)^2}{4U^2 V} + \frac{\partial_r U}{UVr} + \frac{\partial_r^2 U}{2UV} - \frac{(\partial_r U)^2}{4U^2 V} - \frac{\partial_r U\partial_r V}{4UV^2} \\
&- \frac{\partial_r V}{V^2 r} + \frac{\partial_r U}{rUV} - \frac{\partial_r V}{rV^2} - \frac{2}{r^2} + \frac{2}{r^2 V} \\
&= \frac{\partial_r^2 U}{UV} - \frac{\partial_r U\partial_r V}{2UV^2} - \frac{(\partial_r U)^2}{2U^2 V} + \frac{2\partial_r U}{UVr} \\
&- \frac{2\partial_r V}{V^2 r} - \frac{2}{r^2}(1 - \frac{1}{V})
\end{aligned}
$$

We can now substitute the appropriate quantities into the Einstein field equations. Since we are concerned with the geometry of spacetime outside of the massive body or the source of curvature, the energy momentum tensor vanishes. The four equations we want to solve are,

$$R_{00} - \frac{1}{2}g_{00}R = 0 \tag{78}$$

$$\begin{aligned}\Rightarrow \frac{-\partial_r^2 U}{2V} + \frac{\partial_r U \partial_r V}{4V^2} + \frac{(\partial_r U)^2}{4UV} - \frac{\partial_r U}{Vr} \\ + \frac{U}{2}(\frac{\partial_r^2 U}{UV} - \frac{\partial_r U \partial_r V}{2UV^2} - \frac{(\partial_r U)^2}{2U^2 V} + \\ \frac{2\partial_r U}{UVr} - \frac{2\partial_r V}{V^2 r} - \frac{2}{r^2}(1 - \frac{1}{V})) = 0 \\ \Rightarrow \frac{\partial_r V}{V^2 r} + \frac{(1 - \frac{1}{V})}{r^2} = 0\end{aligned}$$

$$R_{11} - \frac{1}{2}g_{11}R = 0 \tag{79}$$

$$\begin{aligned}\Rightarrow \frac{\partial_r^2 U}{2U} - \frac{(\partial_r U)^2}{4U^2} - \frac{\partial_r U \partial_r V}{4UV} - \frac{\partial_r V}{Vr} \\ - \frac{V}{2}(\frac{\partial_r^2 U}{UV} - \frac{\partial_r U \partial_r V}{2UV^2} - \frac{(\partial_r U)^2}{2U^2 V} + \frac{2\partial_r U}{UVr} \\ - \frac{2\partial_r V}{V^2 r} - \frac{2}{r^2}(1 - \frac{1}{V})) = 0 \\ \Rightarrow \frac{-\partial_r U}{UVr} + \frac{(1 - \frac{1}{V})}{r^2} = 0\end{aligned}$$

$$R_{22} - \frac{1}{2}g_{22}R = 0 \tag{80}$$

$$\begin{aligned}\Rightarrow \frac{r\partial_r U}{2UV} - \frac{r\partial_r V}{2V^2} - 1 + \frac{1}{V} \\ - \frac{r^2}{2}(\frac{\partial_r^2 U}{UV} - \frac{\partial_r U \partial_r V}{2UV^2} - \frac{(\partial_r U)^2}{2U^2 V} + \frac{2\partial_r U}{UVr} \\ - \frac{2\partial_r V}{V^2 r} - \frac{2}{r^2}(1 - \frac{1}{V})) = 0 \\ \Rightarrow \frac{-\partial_r U}{U} + \frac{\partial_r V}{V} - \frac{r\partial_r^2 U}{U} + \frac{r\partial_r U \partial_r V}{2UV} + \frac{r(\partial_r U)^2}{2U^2} = 0\end{aligned}$$

$$R_{33} - \frac{1}{2}g_{33}R = 0 \tag{81}$$

$$\Rightarrow \sin^2\theta R_{22} - \frac{r^2\sin^2\theta}{2}R = 0$$
$$\Rightarrow R_{22} - \frac{r^2}{2}R = 0$$

We can now solve for U and V and obtain the full general form of the Schwarzchild metric. We first solve for V through equation (54) which can be written as (after multiplying by V and dividing by r),

$$\frac{\partial_r V}{V} + \frac{(V-1)}{r} = 0$$
$$\frac{\partial_r V}{V(V-1)} + \frac{1}{r} = 0$$
$$\frac{dV}{V(V-1)} = \frac{-dr}{r}$$

Integrating this,

$$\ln\frac{1}{r} + C = \ln\frac{(V-1)}{V}$$
$$\frac{C}{r} = \frac{V-1}{V}$$
$$V(r) = \frac{1}{1-\frac{C}{r}}$$

for some constant C. To find U, we insert our expression for V into equation (55) which can be rewritten as (after multiplying by V and r),

$$\frac{-\partial_r U}{U} + \frac{(V-1)}{r} = 0$$
$$\frac{-\partial_r U}{U} + \frac{(\frac{1}{1-\frac{C}{r}} - 1)}{r} = 0$$
$$\frac{-\partial_r U}{U}(1-\frac{C}{r}) + \frac{C}{r^2} = 0$$
$$\frac{\partial_r U}{U} = \frac{C}{r^2 - Cr}$$
$$\frac{dU}{U} = \frac{Cdr}{r^2 - Cr}$$

We integrate both sides again to get,

$$U(r) = (1-\frac{C}{r}) \tag{82}$$

Thus, the general form of our metric is,

$$ds^2 = -(1-\frac{C}{r})dt^2 + (1-\frac{C}{r})^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{83}$$

Indeed, this metric is spherically symmetric by construction. In fact, a theorem due to Birkhoff states that any spherically symmetric spacetime of the vacuum field equations is isometric to the Schwarzchild spacetime. Since we want our metric to be asymptotically flat as $m \to 0$, we hypothesize that the constant C is proportional to m. Noting that we must regain Newtonian physics in the low mass limit, we can deduce that $C = \frac{2Gm}{c^2}$ where we let $G = c = 1$. The complete Schwarzchild metric is thus,

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{84}$$

Since the Schwarzchild metric describes the spacetime manifold M outside some spherical body, we represent this as the region $r > 2m$. On first glance, the metric seems to have singularities at $r = 0$ and $r = 2m$. But the next question to ask is whether these points are actually singularities i.e. whether the manifold can be extended past $r > 2m$. There in fact does exist some larger manifold M' such that M is embedded in M' and has the Schwarzchild metric in $r > 2m$. The obvious place to try to extend the manifold is at $r = 2m$. We have good reason to believe that there is no singularity at $r = 2m$, because the calculation of the Krestchmann invariant $K = R^{abcd}R_{abcd}$ shows that K does not diverge as $r \to 2m$ as $K \sim \frac{m^2}{r^6}$. Thus, in order to confirm that M can be extended, we need to find some set of nice coordinate transformations. This is indeed possible, as we first define the coordinate r^* by an integral,

$$\frac{dr^*}{dr} \equiv 1 + \frac{2m}{r - 2m} \tag{85}$$

and the advanced and retarded null coordinates,

$$v \equiv t + r^*$$
$$w \equiv t - r^*$$

Under the advanced coordinate v, we can eliminate t from the original metric since,

$$t = v - r^*$$
$$dt = dv - \frac{dr}{1 - \frac{2m}{r}}$$

Substituting this into the metric ,

$$ds^2 = -(1 - \frac{2m}{r})dv^2 + 2dvdr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{86}$$

These are called the ingoing Eddington-Finkelstein coordinates. In this coordinate system, the metric is in fact non-singular at $r = 2m$. Evidence supporting this is that the metric has determinant $-r^4 \sin^2\theta$ and is non-degenerate for any $r > 0$, in particular for $r = 2m$. The inverse metric is also well behaved. Importantly, this metric extends the Schwarzchild solution past $r = 2m$ to the larger manifold M' defined in the region $0 < r < \infty$. It is still a solution of the field equations, since the components are real analytic functions for all $r > 0$. This uses the fact that if analytic functions satisfy the field equations on some strictly smaller open set ($2m < r < \infty$), then they hold everywhere.

One thing to note is that the surface $r = 2m$ is a null surface in M'. This can be seen by taking the surfaces of constant r. Then, the 1-form $n = dr$ is a normal to each such surface. Under the incoming Eddington-Finkelstein coordinates,

$$g^{\mu\nu} n_\mu n_\nu = 1 - \frac{2m}{r} \tag{87}$$

and hence the above is zero if and only if $r = 2m$. This null surface acts as a one-way membrane, letting future-directed timelike and null curves cross only from the region $r > 2m$ to the $r < 2m$. These curves will approach $r = 0$ in a finite affine distance. The Krestchmann invariant ($\sim \frac{m^2}{r^6}$) suggests that $r = 0$ is indeed a real singularity. This implies that the Schwarzchild spacetime cannot be extended further past $r = 0$.

One can similarly obtain the outgoing Eddington-Finkelstein coordinates by using the coordinate w instead of v. These coordinates are similar to the ingoing coordinates, except they reverse the direction of time. If one denotes the extension given by the outgoing coordinates by M'', there in fact exists a larger manifold M^* in which both M' and M'' are imbedded. To obtain it, one uses coordinates given by Kruskal,

$$\begin{aligned} x' &= \frac{1}{2}(v' - w') \\ t' &= \frac{1}{2}(v' + w') \end{aligned}$$

where $v' = \exp(\frac{v}{4m})$ and $w' = -\exp(\frac{-w}{4m})$. The metric given by Kruskal is,

$$ds^2 = F^2(t', x')(-dt'^2 + dx'^2) + r^2(t', x')(d\theta^2 + r^2 \sin^2\theta d\phi^2) \tag{88}$$

where $F^2 = \exp(\frac{-r}{2m})\frac{16m^2}{r}$ and r is defined implicitly by $t'^2 - x'^2 = -(r - 2m)\exp(\frac{r}{2m})$. The Penrose diagram for the maximal extension M^* is shown below

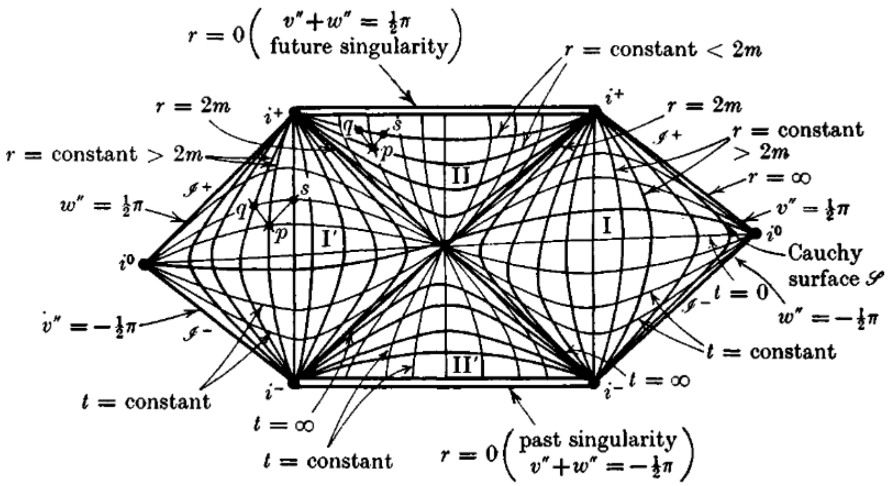

Figure 8: Penrose diagram of Schwarzchild spacetime in Kruskal coordinates. [3], pg.154

We see that there are four regions I, II, I', II'. Region I is given by $x' > |t'|$, and is isometric to the region of the Schwarzchild spacetime for which $r > 2m$. The region composed of I and II is given by $x' > -t'$, and is isometric to M', the manifold given by the incoming Eddington-Finkelstein coordinates. The region composed of I and II' given by $x' > t'$ is isometric to M'', the manifold given by the outgoing Eddington-Finkelstein coordinates. Finally, there is the region I', which is isometric to the same space as region I. Similar geometries can be drawn between the above Penrose diagram and that of Minkowski space, especially in regions I and I'.

The Kruskal extension M^* is the unique analytic and locally inextendible extension of Schwarzchild spacetime. Examining the behavior of geodesics gives the starting point in the study of black holes. If we consider the future light cone of any point outside of the surface $r = 2m$, the radially outwards null geodesics will reach timelike infinity, but the ones pointing radially inwards will eventually reach the singularity at $r = 0$. However, if the observer lies within $r = 2m$, then the null geodesics of the future light cone will inevitably hit the singularity. Thus, once one passes the level set $r = 2m$, it becomes impossible to avoid the singularity. The surface $r = 2m$ is known as the event horizon of the black hole, and marks the point of no return once an observer passes it.

4.5 Reissner-Nordstrom spacetime

The Reissner-Nordstrom solution describes spacetime outside a spherically symmetric electricaly charged body. The energy momentum tensor is thus that of the electromagnetic field. The metric is given by,

$$ds^2 = -(1 - \frac{2m}{r} + \frac{e^2}{r^2})dt^2 + (1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (89)$$

where m is the mass, and e is the electric charge of the body. This solution has little application physically, since massive bodies will likely be electrically neutral; if they are electrically charged, they will attract oppositely charged particles that will then neutralize their charge. Although similarities can be drawn between (65) and the Schwarzchild metric, the main difference is that (65) has coefficients that are quadratic in r. Thus, when calculating possible singularities of the metric, there will in fact be two possible event horizons. Singularities may occur at,

$$\begin{aligned} 1 - \frac{2m}{r} + \frac{e^2}{r^2} &= 0 \\ r^2 - 2mr + e^2 &= 0 \\ &\Rightarrow r_{\pm} = m \pm (m^2 - e^2)^{\frac{1}{2}} \end{aligned}$$

There are three possible cases corresponding to $e^2 > m^2$, $e^2 \le m^2$, and $e^2 = m^2$. For $e^2 > m^2$, r will be imaginary, and hence the metric will be non-singular everywhere except for the irremovable singularity $r = 0$. For $e^2 \le m^2$, the metric will have coresponding singularities at r_+ and r_- and is thus defined in the regions $\infty > r > r_+$, $r_+ > r > r_-$, and $r_- > r > 0$. If $e^2 = m^2$, $r_+ = r_-$ and the metric will just be defined in $\infty > r > r_+$, and $r_+ > r > 0$.

Like in the Schwarzchild case, we can remove these apparent singularities through a nice change of coordinates and obtain a maximal extension,

$$r^* = \int \frac{dr}{1 - \frac{2m}{r} + \frac{e^2}{r^2}} \quad (90)$$

and the advanced and retarded coordinates $v = t + r^*$, $w = t - r^*$, The metric (65) will then take the form,

$$ds^2 = -(1 - \frac{2m}{r} + \frac{e^2}{r^2})dvdw + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (91)$$

The case for $e^2 \le m^2$ is already maximally extended, since the metric is defined everywhere except $r = 0$. In the case of $e^2 < m^2$, we can define,

$$\begin{aligned} v'' &= \arctan(\exp(\frac{r_+ - r_-}{4r_+^2}v)) \\ w'' &= \arctan(-\exp(\frac{-r_+ + r_-}{4r_+^2}w)) \end{aligned}$$

and thus under these coordinates,

$$ds^2 = (1 - \frac{2m}{r} + \frac{e^2}{r^2})64\frac{r_+^4}{(r_+ - r_-)^2}\csc 2v'' \csc 2w'' dv'' dw'' + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad (92)$$

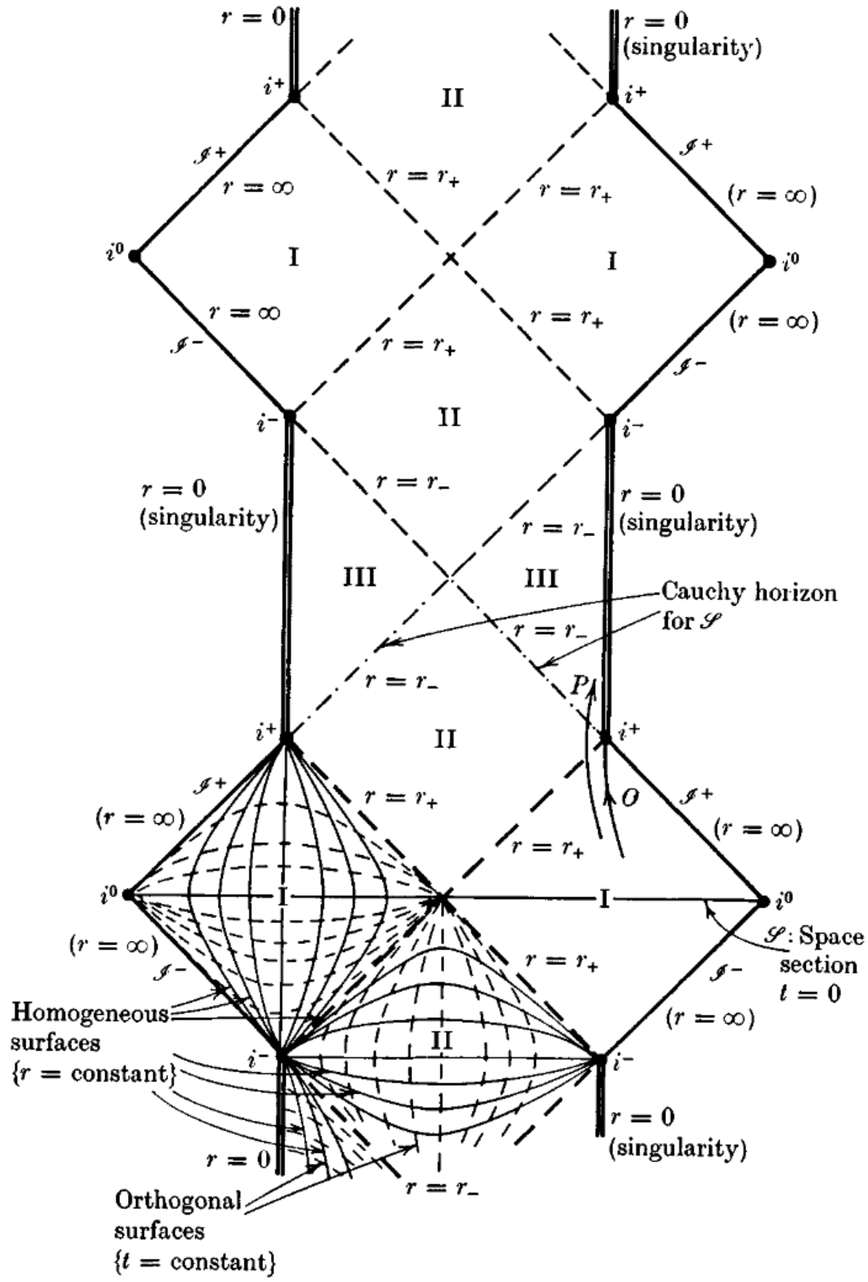

Figure 9: Penrose diagram of the maximally extended Reissner-Nordstrom spacetime where $e^2 < m^2$. [3], pg.158

Using (92), we obtain a maximal extension of the spacetime analytic at all points except at $r = r_+$, where it is at least C^2. From the Penrose diagram, the maximal extension consists of an infinite chain of three regions I, II, and III, where regions II and III subsequently lie between region I, which is Minkowskian. There is the irremovable singularity at $r = 0$; however unlike in Schwarzchild spacetime, the singularity is timelike. Thus, future directed timelike curves crossing between the event horizons $r = r_+$ and $r = r_-$ in regions II and III can avoid $r = 0$, which is fundamentally different than in the Schwarzchild spacetime. Hence, observers could theoretically travel from region I to the subsequent region I, which invites the intriguing possibility of faster-than light travel.

5 Bibliography

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