COLLECTIVE COORDINATES AND KONTSEVICH-SOIBELMAN WALL CROSSING FORMULA

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Contents

1. Collective Coordinates

This was a talk I gave on 11/4/22 for Eric's class on branes, moduli, etc.

1.1. **Introduction.** Collective coordinates expansion is a method frequently used to study dynamical properties of BPS solitons. In [Gau93], collective coordinates are used to show that dynamics in the semiclassical limit (or when low-energy is considered) of monopoles in $N = 2$ supersymmetric Yang-Mills-Higgs theory are determined by an effective $N = 4$ supersymmetric quantum mechanics. The idea of collective coordinates is to incorporate time dependency of fields as coordinates of the moduli space. The moduli space of BPS monopoles has an explicit metric inherited from Euclidean space and can be shown to have a hyperkähler structure. The latter endows the theory with further supersymmetries. The trajectories of BPS monopoles are along geodesics of the inherited metric. We first review Yang-Mills-Higgs theory, its supersymmetric extension, and then turn to collective coordinate dynamics.

1.2. **BPS monopoles.** Recall that the original setting was $d = 3, N = 4$ Yang-Mills Higgs theory. Let A be an $SO(3)$ connection and Φ be a Higgs field with values in the adjoint bundle, i.e. $A \in \Omega^1(M, \mathfrak{so}(3))$ and $\Phi \in \Omega^0(M, \text{ad P})$. Denote $F \in \Omega^2(M, \mathfrak{so}(3))$ and ∇ be the covariant derivative associated to A, i.e. $\nabla_m = \partial_m + [A_m, \cdot]$. The Yang-Mills-Higgs action is the following functional,

$$\mathcal{S}^{YMH}(A) = \int_M d^3x \frac{-1}{4}\text{Tr } F^{mn}F_{mn} + \frac{1}{2}\text{Tr } \nabla^\mu\Phi\nabla_\mu\Phi \tag{1.1}$$

It is convenient to work in the $A_0 = 0$ gauge or imposing Gauss's law. We apply this constraint to the other components of A as well,

$$\nabla_i \dot{A}_i + [\Phi, \dot{\Phi}] = 0$$

In this gauge, the Lagrangian is given by $L = T - V$, where the kinetic energy T is given by

$$T = \frac{1}{2}\int_M d^3x Tr(\dot{A}_i\dot{A}_i + \dot{\Phi}\dot{\Phi})$$

and the potential energy V is of the form

$$V = \frac{1}{2}\int_M d^3x Tr(B_iB_i + \nabla_i\Phi\nabla_i\Phi)$$

where $B = \frac{1}{2}\epsilon_{ijk}F_{jk}$ is the non-abelian field strength. Note that $L = L(t)$ as the fields are time dependent. To construct static monopole solutions, we minimize the static energy V, which Bogomol'nyi showed can be rewritten as

$$V = \int d^3x Tr[\frac{1}{2}(B_i \mp \nabla_i\Phi)(B_i \mp \nabla_i\Phi)] \pm 4\pi k$$

where $k = \frac{1}{4\pi}\int d^3x\partial_i Tr(B_i\Phi)$ is the monopole number, which is an integer topological invariant. We have the bound

$$V \geq 4\pi|k|$$

We see that V is minimized when the bound is saturated or equivalently

$$B_i = \pm\nabla_i\Phi \tag{1.2}$$

which are called the Bogomol'nyi equations. The solutions of these equations are called *BPS monopoles*.

1.3. **Moduli space of BPS monopoles.** We lift to $d = 4$ by using the fact that self-dual Yang-Mills on $\mathbb{R}^4$ with x^4-translational invariance is equivalent to the Bogomol'nyi equations on $\mathbb{R}^3$. Let $\mathcal{A} = \{W_\mu(x)\}, x \in \mathbb{R}^4$ denote the space of finite energy field configurations. To obtain x^4-translational invariance, we set the x^4 direction of W_μ to be the Higgs field. Thus, the components of W_μ are,

$$W_i = A_i, W_4 = \Phi$$

Let $\mathcal{G}$ denote the gauge group. Then, the configuration space of the BPS system is given by

$$\mathcal{C} = \mathcal{A}/\mathcal{G}$$

Let $\dot{W}, \dot{V}$ be two tangent vectors on $\mathcal{A}$. A natural metric on $\mathcal{A}$ is induced by the Euclidean metric on $\mathbb{R}^4$

$$\mathcal{G}(\dot{W}, \dot{V}) = \int d^3x Tr(\dot{W}_\mu\dot{V}_\mu)$$

The choice of gauge $A_0 = 0$ or Gauss's law says that the tangent vectors to $\mathcal{C}$ are orthogonal to the gauge orbits.

$$\nabla_\mu \dot{W}_\mu = 0$$

This implies the metric on $\mathcal{A}$ descends to the quotient $\mathcal{C}$.

Let $\mathcal{M}_k \subset \mathcal{C}$ be the moduli space of k-monopoles, that is monopole solutions of the Bogomol'nyi equation with monopole number k. Let $\{X^a\}, a = 1, \ldots, \dim \mathcal{M}_k$ be coordinates on the moduli space. The tangent vectors to $\mathcal{M}_k$ satisfy linearized Bogomol'nyi equations,

$$\nabla_{[\mu} \dot{W}_{\nu]} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \nabla_\rho \dot{W}_\sigma$$

The coordinates X^a will serve as a subset of our collective coordinates.

The moduli space $\mathcal{M}_k$ was shown to be hyperkähler by Atiyah and Hitchin. Indeed, the three almost complex structures from those on $\mathbb{R}^4$ (endowed with quaternionic structure) can be shown to descend to $\mathcal{M}_k$. These almost complex structures will be used to define supersymmetry transformations later on. We denote these almost complex structures on $\mathcal{M}_k$ by $\tilde{J}^{(m)}$, $m = 1, 2, 3$.

1.4. **Collective Coordinate Expansion.** So far, we have discussed the purely bosonic Yang-Mills-Higgs theory. It is possible to extend it to an N=2 supersymmetric theory via the following supersymmetric Lagrangian density

$$\begin{aligned}\mathcal{L} = Tr\{&\frac{-1}{4}(F_{mn})^2 + \frac{1}{2}(\nabla_m P)^2 + \frac{1}{2}(\nabla_m S)^2 - \frac{1}{2}([S, P])^2 \\ &+ i\bar{\chi}\gamma^m \nabla_m \chi - \bar{\chi}\gamma_5 [P, \chi] - i\bar{\chi}[S, \chi]\}\end{aligned} \tag{1.3}$$

where P, S are Higgs fields, and χ is a Dirac fermion.

Fluctuations about a monopole solution $W_\mu(x)$ may contain massive modes along with zero modes. In the purely bosonic theory, zero modes were closely described by tangent vectors to $\mathcal{M}_k$. In the supersymmetric theory, the fermionic zero modes are given by time independent solutions to the Dirac equation in the presence of a monopole.

$$i\gamma^i \nabla_i \chi - i[\Phi, \chi] = 0$$

If we focus on low energy dynamics, then we ignore the massive modes, and only introduce collective coordinates for each zero mode. For the bosonic collective coordinates, we take the coordinates $\{X^a\}$ of the moduli space. The fermionic collective coordinates we denote by λ^a, which are one component Grassmann odd objects. We study the $N = 2$ supersymmetric action under the following ansatz

$$\begin{aligned} W_\mu(x, t) &= W_\mu(x, X(t)) \\ \chi &= \delta_a W_\mu \Gamma^\mu \epsilon_+ \lambda^a(t) \end{aligned} \tag{1.4}$$

with λ^a satisfying

$$-i\lambda^a (J^{(3)})_a^b = \lambda^b$$

The idea of collective coordinates is we incorporate time dependency as coordinates in some moduli space, that usually has extra symmetries, such as a HK structure. If we plug in the above ansatz into the action. Denote by n_∂ the number of time derivatives and n_f the number of fermions, collective coordinates expands in $n = n_\partial + \frac{1}{2}n_f$. If we substitute the ansatz (3.4) into the action (3.3), we obtain an action of order $n = 2$. To obtain a consistent expansion, we must ensure the ansatz solves the equations of motion to order $n = 0, \frac{1}{2}, 1$. To order $n = 0, \frac{1}{2}$, (3.4) solves the equations of motion trivially. To order $n = 1$, the ansatz must be supplemented by

$$\begin{aligned} A_0 &= \dot{X}^a \epsilon_a - i\phi_{ab}\lambda^{\dagger a}\lambda^b \\ P &= i\phi_{ab}\lambda^{\dagger a}\lambda^b \end{aligned}$$

After substituting the ansatz into the action (3.3) and integrating over spatial degrees of freedom, one obtains the following effective action

$$S_{eff} = \frac{1}{2}\int dt \mathcal{G}_{ab}\{\dot{X}^a\dot{X}^b + 4i\lambda^{\dagger a}D_t\lambda^b\} - 4\pi k \tag{1.5}$$

where the covariant derivative on the fermionic coordinates λ^a is defined as

$$D_t\lambda^b = \dot{\lambda}^b + \Gamma^b_{ac}\dot{X}^a\lambda^c$$

and Γ^b_{ac} are the Christoffel symbols associated to the metric $\mathcal{G}_{ab}$. Notice that the effective action is a supersymmetric quantum mechanics ($d = 1$) action. Since the metric on $\mathcal{M}_k$ is hyperkähler, the effective action is invariant under $N = 4$ worldline supersymmetry,

$$\begin{aligned} \delta X^a &= i\beta_4\psi^a + i\beta_m\psi^b(J^{(m)})^a_b \\ \delta\lambda^a &= \frac{-\dot{X}^a\beta_4 + \beta_m\dot{X}^b(J^{(m)})^a_b}{\sqrt{2}} \end{aligned}$$

where β_m are four worldline parameters which are odd Grassmann real. These are the unbroken supersymmetries of the field theory. Writing

$$\begin{aligned} \delta W_\mu &= \delta X^a\delta_a W_\mu \\ \delta\chi &= \delta_a W_\mu \Gamma^\mu\epsilon_+\delta\lambda^a + s_a(\delta W_\mu)\Gamma^\mu\epsilon_+\lambda^a \end{aligned}$$

and applying supersymmetries of the Yang-Mills-Higgs action, one can further relations of the worldline parameters β_m. In conclusion, (3.5) shows that low energy dynamics is described by an $N = 4$ supersymmetric quantum mechanics.

2. Wall Crossing

2.1. **Vanilla BPS wall crossing.** Recall some basics about BPS states, following [Neit10]. The Hilbert space of the quantum theory is graded by a charge lattice

$$\mathcal{H} = \oplus_{\gamma\in\Lambda}\mathcal{H}_\gamma$$

(The lattice could be $K_0(D^bCoh)$ or $H_2(X,\mathbb{Z})$)One adds in supersymmetry, or fermions and bosons, by introducing a $\mathbb{Z}/2$-graded Lie algebra $\mathcal{A}$ with generators and commutators, and considers $\mathbb{Z}/2$-graded representations of $\mathcal{A}$ on $\mathcal{H}$. We in particular consider two elements of $\mathcal{A}$ that act as scalars in any representation of $\mathcal{A}$; we have the Casimir element in the universal enveloping algebra whose square root gives the mass of particles, and the central charge Z, defined as one of the generators of $\mathcal{A}$. It can be shown via a simple computation of commutators that the *BPS bound* holds

$$M \geq |Z|$$

in any representation. *Short* representations of $\mathcal{A}$, or the BPS particles, are those which saturate the BPS bound, and *long* representations are those which satisfy $M > |Z|$. There is an index, the trace of the Witten operator, that counts the BPS particles of the theory with multiplicity.

$$\Omega(\gamma) = \#\text{ of BPS particles with multiplicity}$$

As an index, it should be invariant under deformation. Notice that it is dependent on the charge vector $\gamma \in \Lambda$. Suppose we have a system of two particles with rest masses M_1 and M_2. By switching to the rest frame, it can be shown that the total mass M of the physical system satisfies the inequality

$$M \geq M_1 + M_2$$

Combined with the BPS bound and triangle inequality, we have

$$M \geq M_1 + M_2 \geq |Z_1| + |Z_2| \geq |Z_1 + Z_2| = M_{BPS}$$

We see that equalities hold when $Z_2 = cZ_1$ for some $c \in \mathbb{R}_{>0}$ and both particles are BPS. Therefore, we see that we run into an issue when counting BPS 2-particle states; there will be contributions from the 1-particle BPS states. Thus, the index is well-defined or provides an invariant count except when $Z_2 = cZ_1$ for some $c \in \mathbb{R}_{>0}$ and $\gamma = \gamma_1 + \gamma_2$. This condition defines a codimension 1 *wall* in some space of deformations, and gives rise to wall crossing.

3. Kontsevich-Soibelman Wall Crossing Formula

3.1. **Introduction.**

Wall crossing has appeared in many contexts including counts of holomorphic discs in a Lagrangian fibration, quiver Donaldson-Thomas invariants, Gromov-Witten theory of blow ups of toric surfaces, counting of geodesics of quadratic differential on a curve, and $N = 2, d = 4$ supersymmetric gauge theory. In many cases, the invariants have been shown to satisfy/follow from the Kontsevich-Soibelman wall crossing formula (WCF),

$$\prod_{\frac{a}{b}\text{ decreasing}} f_{a,b} = \prod_{\frac{a}{b}\text{ increasing}} f_{a,b}$$

Here, the $f_{a,b}$ are automorphisms indexed by $(a,b) \in \mathbb{Z}^2$ and products on the LHS (RHS) are ordered such that $\frac{a}{b}$ is decreasing (increasing). Intuitively, one can imagine the LHS of the formula is a set of rays of decreasing slope heading towards a singularity and the RHS of the formula is the set of rays of increasing slope scattering out. Computing the above identity explicitly is often quite difficult, and has only been done in a few cases. We will give a few examples of these computations, and various contexts in which the WCF has lended inspiration. First, we define the automorphisms $f_{a,b}$ of interest. They will be automorphisms of the algebraic torus. We specify to $d = 2$.

3.2. **The Algebraic Torus.** Let $T = (\mathbb{C}^*)^2$ be the two dimensional algebraic torus. The set of its characters $M = Hom(T, \mathbb{C}^*)$ is a two dimensional lattice since each character is described by $(x, y) \mapsto x^a y^b$ for some $m = (a, b) \in \mathbb{Z}^2$. The algebra of regular functions on the torus $\Gamma(\mathcal{O}_T)$ has as basis the lattice of characters, i.e.

$$\Gamma(\mathcal{O}_T) = \bigoplus_{m \in M} \mathbb{C} z^m = \mathbb{C}[M]$$

with multiplication defined as $z^m \cdot z^{m'} = z^{m+m'}$. Choose an orientation on M, i.e. an integral, unimodular, skew-symmetric, bilinear form $\langle , \rangle$ so that $\bigwedge^2 M \cong \mathbb{Z}$. Define a bracket on $\Gamma(\mathcal{O}_T)$ by,

$$\{z^m, z^{m'}\} = \langle m, m' \rangle z^{m+m'}$$

This makes $\Gamma(\mathcal{O}_T)$ into a Poisson Lie algebra. Denote by Ω by the corresponding algebraic symplectic form. If a basis $\{m_1, m_2\} \subset M$ is chosen such that $\langle m_1, m_2 \rangle = 1$, then denoting $z_1 = z^{m_1}$ and $z_2 = z^{m_2}$, we can identify $M \cong \mathbb{Z}^2, \mathbb{C}[M] = \mathbb{C}[z_1^{\pm 1}, z_1^{\pm 1}]$, and $\Omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$.

For each $m \in M$, define a birational automorphism $f_m \in Aut(\Gamma(\mathcal{O}_T))$ by

$$f_m(z^{m'}) := z^{m'}(1 \pm z^m)^{\langle m, m' \rangle}$$

Notice the similarity with cluster transformations.

3.3. **Quantized Algebraic Torus.** We introduce a formal variable $q = e^{i\hbar}$ to incorporate non-commutativity of the multiplication. Denote $\hat{T} = (\mathbb{C}^*)^2$ be the noncommutative algebraic torus whose algebra of functions is

$$\Gamma(\mathcal{O}_{\hat{T}}) = \bigoplus_{m \in M} \mathbb{C}[q^{\pm \frac{1}{2}}] z^m = \mathbb{C}[q^{\pm \frac{1}{2}}][M]$$

where the multiplication rule is defined by $z^m \cdot z^{m'} = q^{\frac{\langle m, m' \rangle}{2}} z^{m+m'}$. Notice that $xy = qyx$ if $\langle \cdot, \cdot \rangle$ is the chosen to be determinant. It is a quantization in the sense that in the limit $q \to 1$, we recover the algebraic torus $T = (\mathbb{C}^*)^2$.

For each $m \in M$, we can take an automorphism $\hat{f}_m \in Aut(\Gamma(\mathcal{O}_{\hat{T}}))$ of the quantized torus to be conjugation by the quantum dilogarithm $\Psi(\hat{z}^m) = \prod_{k \geq 0} \frac{1}{1 - q^{k + \frac{1}{2}} \hat{z}^m}$, i.e. we take

$$\hat{f}_m = Ad_{\Psi(\hat{z}^m)}$$

It's a fact that $\lim_{q\to 1} \hat{f}_m = f_m$.

3.4. **Wall Structures.** Our *wall crossing structures* or *scattering diagrams* will be a set of codimension 1 walls, with automorphisms of the algebraic torus, with or without quantization, attached to each wall. The structure is called *consistent* if for any loop in the structure, the composition of automorphisms is the identity. Computing consistent structures becomes complex very quickly, and to compute a consistent diagram in practice requires inductively computing to a certain order k and increasing $k \to \infty$. In theory, fortunately Kontsevich-Soibelman showed that an initial wall structure determines a unique consistent structure, and consistent structures can be obtained by only adding rays or half lines.

Example 3.1. The elementary example of scattering when $\langle m, m' \rangle = 1$ (taking $m = (1,0)$). The incoming rays are $(\mathbb{R}_{\le 0}(1,0), (1+x))$ and $(\mathbb{R}_{\le 0}(0,1), (1+y))$. Consistency is obtained by propagating the two incoming rays and adding a third ray, i.e. adding $(\mathbb{R}_{\ge 0}(1,0), (1+x))$, $(\mathbb{R}_{\ge 0}(1,0), (1+x))$, $(\mathbb{R}_{\ge 0}(1,1), (1+xy))$. Notice the incoming rays are of decreasing slope, and the outgoing rays are of increasing slope. One can check by hand that

$$f_{1,0} f_{0,1} = f_{0,1} f_{1,1} f_{1,0}$$

which is in accordance with WCF. Physically, two BPS particles cross the wall and split off a new third one.

Example 3.2. When $\langle m, m' \rangle = 2$ (taking $m = (2,0)$), the number of outgoing rays is infinite. The incoming rays are $(\mathbb{R}_{\le 0}(1,0), (1+x))$ and $(\mathbb{R}_{\le 0}(0,1), (1+y)^2)$. Consistency is obtained by adding the following outgoing rays on the RHS

$$f_{1,0} f_{0,2} = f_{0,2} f_{1,4} f_{2,6} \cdots f_{1,2}^{-2} \cdots f_{3,4} f_{2,2} f_{1,0}$$

Notice the slopes a/b decrease on the LHS, and increase on the RHS. The factors (exponents) on the RHS are correspond to the BPS spectrum of $N = 2, d = 4$ super Yang-Mills studied by Seiberg and Witten. They are also related to counts of geodesics and saddle connections studied on curves by Neitzke.

Example 3.3. When $\langle m, m' \rangle = 3$ (taking $m = (3,0)$), the outgoing walls and their automorphisms are not known. There is a region in which the walls are dense.

3.5. **Aside to Stability Scattering Diagrams.** Bridgeland defines scattering diagrams associated to 2-acyclic quiver Q, in which each indecomposable representation of the Q defines a wall in a space of semi-slope stability conditions of Q. The scattering diagrams in Examples 5.1, 5.2, 5.3 correspond to that of the A_2 quiver, Kronecker quiver, and generalized Kronecker quiver with three arrows.

4. Some results motivated from wall crossing

4.1. **Quiver DT and Cohomological Hall Algebra.** Let $Q = (V, E)$ be a directed quiver. What is the moduli of its representations? Let $d \in \mathbb{N}_{>0}^{|V|}$ be the dimension vector of Q, with d_i the dimension of vector space V_i. For each directed edge $i \to j$, there is a moduli $Hom(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$ of linear maps. By summing over all directed edges, we define

$$R_d := \bigoplus_{\exists \text{ edge } i \to j} Hom(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

The gauge group $G_d := \prod_{V_i} GL_{d_i}(\mathbb{C})$ acts on R_d by basis change at each vertex. Therefore, we can take the equivariant cohomology of R_d, i.e. $H^*_{G_d}(R_d)$ and hence its equivariant Euler characteristic $\chi_{G_d}(R_d)$.

Let's suppose we specialize to a quiver Q with just one node and m-loops where $m \geq 1$. The dimension vector d here is a positive integer $d > 0$. Note that the COHA of an m-loop quiver with one node is the non-commutative Hilbert scheme for the free $\mathbb{C}-$algebra in m variables in [Rei11]. We form the generating series $F(t)$ of the equivariant Euler characteristics of R_d, i.e.

$$F(t) := \sum_{d \in \mathbb{N}_{>0}^{|V|}} \chi_{G_d}(R_d) t^d$$

where t is a variable keeping track of d. Motivated by Konsevich-Soibelman, the Donaldson-Thomas invariants of the quiver are defined to be the rational numbers $DT_d^{(m)} \in \mathbf{Q}$ in the following expression

$$F((-1)^{m-1}t) = \prod_{d \geq 1} (1 - t^d)^{-(-1)^{(m-1)d} d DT_d^{(m)}}$$

These numbers are well-defined because F is a integral power series with constant term 1. It is proven in [Rei09] that remarkably $DT_d^{(m)} \in \mathbb{N}$ with an explicit formula

$$DT_d^{(m)} = \frac{1}{d^2} \sum_{n|d} \mu(\frac{d}{n})(-1)^{(m-1)(d-n)} \binom{md-1}{d-1}$$

4.2. **Gromov-Witten Invariants (The Tropical Vertex).** The main idea of the "Tropical Vertex" [GPS09] is that consistency of the scattering diagram is equivalent to computing certain log Gromov-Witten invariants of blow ups of toric surfaces. Without quantization of the algebraic torus, the tropical vertex computes $g = 0$ invariants. With quantization, it computes higher genus invariants [Bou18]. It says the automorphisms of the outgoing rays can be expressed in terms of log Gromov-Witten invariants.

Example 4.1. We take the elementary example of scattering in 5.1, but this time we quantize the algebraic torus. Thus, the incoming rays are $(\mathbb{R}_{\leq 0}(1,0), Ad_{\Psi(\hat{z}^{(1,0)})})$ and

$(\mathbb{R}_{\leq 0}(1,0), Ad_{\Psi(\hat{z}^{(0,1)})})$, with automorphisms $Ad_{\Psi(x)}$. Because the quantum dilogarithm satisfies the Faddev-Kashaev pentagon identity

$$\Psi(\hat{z}^{(1,0)})\Psi(\hat{z}^{(0,1)}) = \Psi(\hat{z}^{(0,1)})\Psi(\hat{z}^{(1,1)})\Psi(\hat{z}^{(1,0)})$$

the consistent diagram is obtained by propagating the two incoming rays and adding a third ray $(\mathbb{R}_{\geq 0}(1,1), Ad_{\Psi(\hat{z}^{(1,1)})})$. We consider the incoming rays and add the single ray $\mathbb{R}_{\geq 0}(1,1)$ to get the fan of $\mathbb{P}^2$. This is the toric surface we will be working with. They way one completes the initial incoming fan does not matter, since all toric varieties are rational and log Gromov-Witten invariants are birationally invariant. Blow up two distinct, non-toric fixed points on the toric boundary. Consider the strict transform of the unique line connecting the two points. The line is rigid, so the only contributions to its Gromov-Witten invariant $N^{Y_m}_{g,(l,l)}$ come from multiple covers.

$$\sum_{g\geq 0} N^{Y_m}_{g,(l,l)} \hbar^{2g-1} = \frac{1}{l}\frac{(-1)^{(l-1)}}{2\sin\frac{l\hbar}{2}}$$

This formula was known in Bryan-Pandharipande, and the quantum tropical vertex recovers it.

4.3. **Gross-Siebert Mirror Symmetry program.** In the Gross-Siebert program, wall crossing structures live in affine manifolds with singularities. One inductively proves consistency of the wall crossing by proving consistency up to a certain order t^k, and then for all k. Consistency is necessary for one to construct the mirror toric degeneration from the dual intersection complex by gluing canonical thickenings.

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